Solutions Manual

PRINCIPLES OF DYNAMICS

second edition

Donald T. Greenwood

Professor, Department of Aerospace Engineering
The University of Michigan

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SOLUTIONS MANUAL

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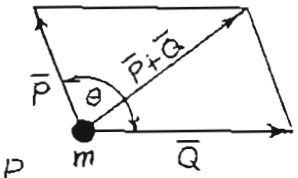
CHAPTER 1

1-1. Let $Q > P$.



$a_{max} = \frac{1}{m}(Q+P)$, $a_{min} = \frac{1}{m}(Q-P)$

$\frac{a_{max}}{a_{min}} = \frac{Q+P}{Q-P} = 3$, giving $Q = 2P$.

$F = ma$
$a = \frac{F}{m}$

Mean acceleration $= \frac{1}{2}(a_{max}+a_{min}) = \frac{Q}{m} = \frac{1}{m}|\bar{P}+\bar{Q}|$

Hence $Q = |\bar{P}+\bar{Q}|$. By cosine law, $|\bar{P}+\bar{Q}|^2 = Q^2+P^2+2QP\cos\theta = Q^2$

Then $P^2 = -2QP\cos\theta = -4P^2\cos\theta$ or $\cos\theta = -1/4$, $\underline{\theta = 104.48°}$

1-2. (a) Auto position $\bar{r}_a = 30t\,\bar{i}$ meters

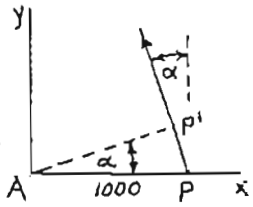
Plane position $\bar{r}_p = 1000\,\bar{i} + 90t\,\bar{j} + 1000\,\bar{k}$ meters

Relative position $\bar{r}_{p/a} = \bar{r}_p - \bar{r}_a = (1000-30t)\,\bar{i} + 90t\,\bar{j} + 1000\,\bar{k}$

Relative velocity $\bar{v}_{p/a} = \frac{d}{dt}(\bar{r}_{p/a}) = \underline{-30\,\bar{i} + 90\,\bar{j}\ \text{m/sec}}$

(b) $AP' = 1000\cos\alpha$, $\alpha = \tan^{-1}\frac{1}{3}$

or $AP' = 1000\,\frac{3}{\sqrt{10}}$



Separation $d_{min} = \sqrt{(AP')^2 + 1000^2}$

$= 1000\sqrt{\frac{9}{10}+1}$

$= \underline{1378.4\ \text{meters}}$

1-3. Given $\bar{e}_1 = l_1\bar{i} + l_2\bar{j} + l_3\bar{k}$

$\bar{e}_2 = m_1\bar{i} + m_2\bar{j} + m_3\bar{k}$

$\bar{e}_3 = n_1\bar{i} + n_2\bar{j} + n_3\bar{k}$

(a) $l_1^2 + l_2^2 + l_3^2 = 1$

$m_1^2 + m_2^2 + m_3^2 = 1$

$n_1^2 + n_2^2 + n_3^2 = 1$

(b)

$$\bar{e}_1 \cdot (\bar{e}_2 \times \bar{e}_3) = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 0$$

(c) Equations of part (a) and

$l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$

$l_1 n_1 + l_2 n_2 + l_3 n_3 = 0$

$m_1 n_1 + m_2 n_2 + m_3 n_3 = 0$

4. $\bar{A} = A_x \bar{i} + A_y \bar{j} + A_z \bar{k}$
$= A_1 \bar{e}_1 + A_2 \bar{e}_2 + A_3 \bar{e}_3$ where $\begin{cases} \bar{e}_1 = \bar{i} \\ \bar{e}_2 = \frac{1}{\sqrt{2}}(\bar{i}+\bar{j}) \\ \bar{e}_3 = \frac{1}{\sqrt{2}}(\bar{i}+\bar{k}) \end{cases}$

equating $\bar{i}$ components,

$A_1 + \frac{A_2}{\sqrt{2}} + \frac{A_3}{\sqrt{2}} = A_x$

rom $\bar{j}$ components, $\frac{A_2}{\sqrt{2}} = A_y$

rom $\bar{k}$ components, $\frac{A_3}{\sqrt{2}} = A_z$

solving,

$A_1 = A_x - A_y - A_z$

$A_2 = \sqrt{2}\, A_y$

$A_3 = \sqrt{2}\, A_z$

-5. Given $[L], [M], [F]$ are fundamental.

$F = ma$, so the dimensions of acceleration $[A] = [M^{-1} F]$

$A] = [L T^{-2}]$ or $[T] = [L^{\frac{1}{2}} A^{-\frac{1}{2}}] = [L^{\frac{1}{2}} M^{\frac{1}{2}} F^{-\frac{1}{2}}]$

-6. Fundamental units: mass — kg
length — ell
time — tee

.et 1 ell = a meters
1 tee = b sec

Density of water $\rho = 1\ g/cm^3 = 1000\ kg/m^3$

Specific wt. $w = \rho g = 9810 \frac{kg}{m^2 sec^2} = 1 \frac{kg}{ell^2\, tee^2}$

Hence $1 = 9810 \left(\frac{ell}{m}\right)^2 \left(\frac{tee}{sec}\right)^2 = 9810\, a^2 b^2$

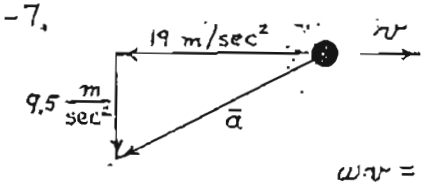
Accel. of gravity $= 9.81 \frac{m}{sec^2} = 1 \frac{ell}{tee^2}$ or $9.81 = \frac{a}{b^2}$

Multiply equations. $9.81 = 9810\, a^3$ giving $a = 0.1$

Then $b = \sqrt{\frac{a}{9.81}} = \frac{1}{\sqrt{98.1}} = 0.1010$

1 ell = 0.1000 meters
1 tee = 0.1010 sec

1-7.

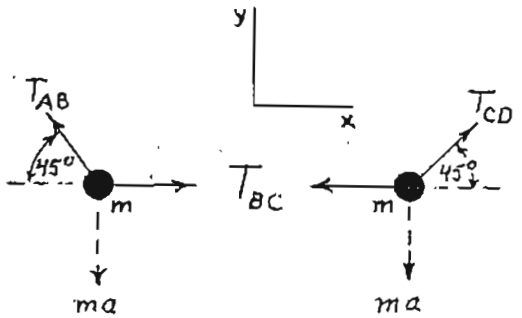


$a = \sqrt{19^2 + 9.5^2} = 21.243 \frac{m}{sec^2}$

For $v = 3000$ m/sec,

$\omega v = 9.5 \frac{m}{sec^2}$, $\omega = \frac{9.5}{3000} = 3.167 \times 10^{-3} \frac{rad}{sec}$

1-8. Use d'Alembert's principle.



Sum y forces on system. $\frac{1}{\sqrt{2}}(T_{AB} + T_{CD}) = 2ma$

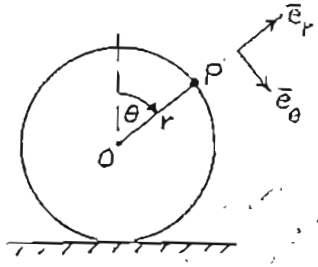
Sum x forces on each particle. $T_{BC} = \frac{T_{AB}}{\sqrt{2}} = \frac{T_{CD}}{\sqrt{2}}$

Then $T_{AB} = T_{CD} = \sqrt{2}\, ma$, $T_{BC} = ma$

CHAPTER 2

2-1. $\omega = \alpha t$ where $\alpha = \text{const.}$

Integrate to obtain $\theta = \frac{1}{2}\alpha t^2$



Acceleration of P is

$$\bar{a} = \bar{a}_O + \bar{a}_{P/O}$$

where the horizontal acceleration $\bar{a}_O = r\alpha(\sin\theta\,\bar{e}_r + \cos\theta\,\bar{e}_\theta)$

P moves in a circular path relative to O.

$$\bar{a}_{P/O} = -r(\alpha t)^2\,\bar{e}_r + r\alpha\,\bar{e}_\theta$$

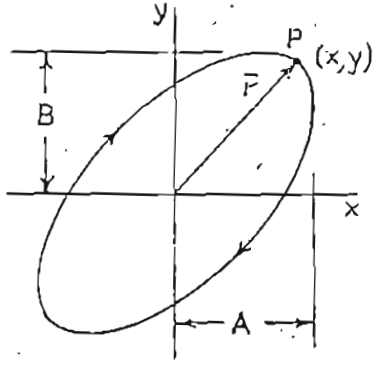
Hence. $\underline{\bar{a} = r\alpha\left[(\sin\theta - \alpha t^2)\,\bar{e}_r + (1+\cos\theta)\,\bar{e}_\theta\right]}$

2-2. Given $x = A\cos\omega t$

$y = B\cos(\omega t + \beta)$

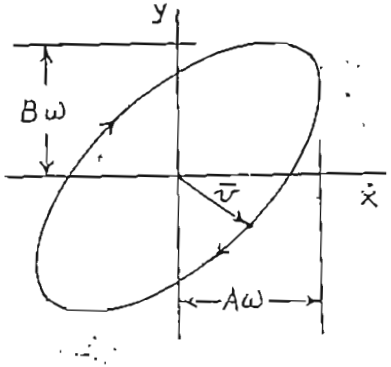
Then $\dot{x} = -A\omega\sin\omega t$

$\dot{y} = -B\omega\sin(\omega t + \beta)$



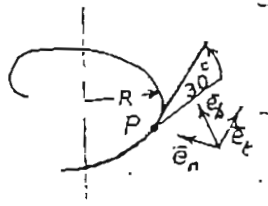
The hodograph is a plot of $\bar{v} = \dot{x}\,\bar{i} + \dot{y}\,\bar{j}$

Note that the two ellipses have the same shape.



2-3. Speed $s = v_0$ and $\dot{s} = a_0$

$$\bar{a} = \dot{s}\,\bar{e}_t + \frac{s^2}{\rho}\,\bar{e}_n$$



From Eq. (2-61), $\rho = R(1+k^2)$

$= R(1+\tan^2 30°) = \frac{4}{3}R$

Then we obtain $\underline{\bar{a} = a_0\,\bar{e}_t + \frac{3v_0^2}{4R}\,\bar{e}_n}$

2nd method: Use the horizontal component of velocity and a circular path of radius R to obtain $a_n = \frac{(v_0\cos 30°)^2}{R} = \frac{3v_0^2}{4R}$

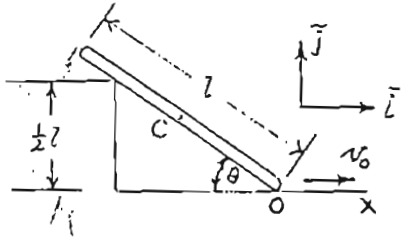
2-4. Position x of lower end O is

$x = \frac{1}{2}l\cot\theta$

giving $\dot{x} = -\frac{1}{2}l\dot{\theta}\csc^2\theta = v_0$

or

$$\underline{\dot{\theta} = -\frac{2v_0}{l}\sin^2\theta}$$



(b) Differentiate again. $\ddot{\theta} = \frac{-4v_0\dot{\theta}}{l}\sin\theta\cos\theta$

Using $\dot{\theta}$ expression, $\underline{\ddot{\theta} = \frac{8v_0^2}{l^2}\sin^3\theta\cos\theta}$

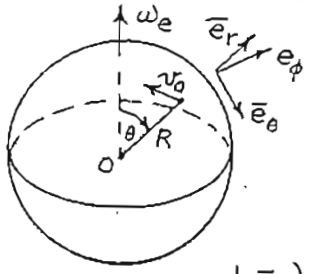
(c) $\bar{v}_C = \bar{v}_O + \bar{v}_{C/O} = v_0\,\bar{i} + \frac{1}{2}l\dot{\theta}(\sin\theta\,\bar{i} + \cos\theta\,\bar{j})$

$= v_0\,\bar{i} + \frac{1}{2}l\left(\frac{-2v_0}{l}\sin^2\theta\right)(\sin\theta\,\bar{i} + \cos\theta\,\bar{j})$

or

$$\underline{\bar{v}_C = v_0\left[(1-\sin^3\theta)\,\bar{i} - \sin^2\theta\cos\theta\,\bar{j}\right]}$$

2-5. Let moving frame rotate with the earth. $\bar\omega = \bar\omega_e$, $\theta = 45^\circ$



$$\bar a = \ddot{\bar R} + \dot{\bar\omega}\times\bar\rho + \bar\omega\times(\bar\omega\times\bar\rho) + (\ddot{\bar\rho})_r + 2\bar\omega\times(\dot{\bar\rho})_r$$

Acceleration of O is $\ddot{\bar R} = 0$

$\dot{\bar\omega} = 0$, $\bar\rho = R\bar e_r$, so $\dot{\bar\omega}\times\bar\rho = 0$

$$\bar\omega\times(\bar\omega\times\bar\rho) = \frac{R}{\sqrt2}\omega_e^2\left(-\frac{1}{\sqrt2}\bar e_r - \frac{1}{\sqrt2}\bar e_\theta\right)$$

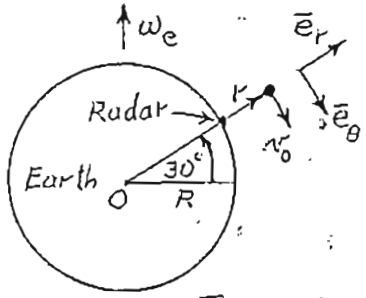
$$\bar\omega = \omega_e\left(\frac{1}{\sqrt2}\bar e_r - \frac{1}{\sqrt2}\bar e_\theta\right)$$

$$(\ddot{\bar\rho})_r = -\frac{v_0^2}{R}\bar e_r \qquad (\dot{\bar\rho})_r = v_0\left(-\frac{1}{\sqrt2}\bar e_\theta - \frac{1}{\sqrt2}\bar e_\phi\right)$$

$$2\bar\omega\times(\dot{\bar\rho})_r = \omega_e v_0\left(-\bar e_r\times\bar e_\theta - \bar e_r\times\bar e_\phi + \bar e_\theta\times\bar e_\phi\right) = \omega_e v_0(\bar e_r + \bar e_\theta - \bar e_\phi)$$

Adding, $\underline{\bar a = \left(-\frac12 R\omega_e^2 - \frac{v_0^2}{R} + \omega_e v_0\right)\bar e_r + \left(-\frac12 R\omega_e^2 + \omega_e v_0\right)\bar e_\theta - \omega_e v_0\bar e_\phi}$

2-6. Rotating frame is fixed in the earth, with origin at O.



$\bar\rho = r\bar e_r$, $\bar\omega = \omega_e\left(\frac12\bar e_r - \frac{\sqrt3}{2}\bar e_\theta\right)$

From (2-102), $\bar v = \dot{\bar R} + (\dot{\bar\rho})_r + \bar\omega\times\bar\rho$.

We know $\bar v = v_0\bar e_\theta$. Solve for $(\dot{\bar\rho})_r$.

$\dot{\bar R} = 0$, $\bar\omega\times\bar\rho = \frac{\sqrt3}{2}r\omega_e\bar e_\phi$, so

$$\underline{(\dot{\bar\rho})_r = v_0\bar e_\theta - \frac{\sqrt3}{2}r\omega_e\bar e_\phi}$$

From (2-106),

$$\bar a = \ddot{\bar R} + \dot{\bar\omega}\times\bar\rho + \bar\omega\times(\bar\omega\times\bar\rho) + (\ddot{\bar\rho})_r + 2\bar\omega\times(\dot{\bar\rho})_r, \qquad \ddot{\bar R} = 0,\ \dot{\bar\omega}\times\bar\rho = 0$$

We know $\bar a = -\frac{v_0^2}{r}\bar e_r$. Solve for $(\ddot{\bar\rho})_r$.

Now $\bar\omega\times(\bar\omega\times\bar\rho) = \frac{\sqrt3}{2}r\omega_e^2\left(-\frac{\sqrt3}{2}\bar e_r - \frac12\bar e_\theta\right) = -\frac34 r\omega_e^2\bar e_r - \frac{\sqrt3}{4}r\omega_e^2\bar e_\theta$

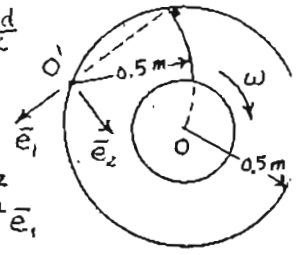
$$2\bar\omega\times(\dot{\bar\rho})_r = 2\omega_e\left(\frac12\bar e_r - \frac{\sqrt3}{2}\bar e_\theta\right)\times\left(v_0\bar e_\theta - \frac{\sqrt3}{2}r\omega_e\bar e_\phi\right)$$
$$= \omega_e v_0\bar e_\phi + \frac{\sqrt3}{2}r\omega_e^2\bar e_\theta + \frac32 r\omega_e^2\bar e_r$$

Hence we obtain

$$\underline{(\ddot{\bar\rho})_r = \left(-\frac{v_0^2}{r} - \frac34 r\omega_e^2\right)\bar e_r - \frac{\sqrt3}{4}r\omega_e^2\bar e_\theta - \omega_e v_0\bar e_\phi}$$

Note that the result is independent of earth's radius R.

*2-7. Moving system has origin at O' and rotates with impeller at $\omega = 40\pi\ \frac{rad}{sec}$



From (2-106),

$$\bar a = \ddot{\bar R} + \dot{\bar\omega}\times\bar\rho + \bar\omega\times(\bar\omega\times\bar\rho) + (\ddot{\bar\rho})_r + 2\bar\omega\times(\dot{\bar\rho})_r$$

$$\ddot{\bar R} = 0.5(40\pi)^2\left(-\frac12\bar e_1 + \frac{\sqrt3}{2}\bar e_2\right) = -3948\bar e_1 + 6838\bar e_2$$

$\dot{\bar\omega} = 0$, $\bar\rho = -0.5\bar e_1$, $\bar\omega\times(\bar\omega\times\bar\rho) = 0.5(40\pi)^2\bar e_1 = 7896\bar e_1$

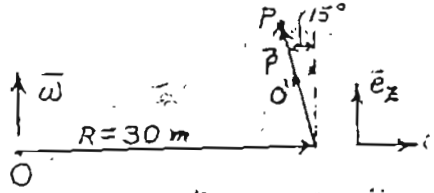
$(\ddot{\bar\rho})_r = \frac{(20)^2}{0.5}\bar e_1 = 800\bar e_1$, $(\dot{\bar\rho})_r = -20\bar e_2$

$2\bar\omega\times(\dot{\bar\rho})_r = -(80\pi)(20)\bar e_1 = -5027\bar e_1$

Adding terms, $\underline{\bar a = -278.7\bar e_1 + 6838\bar e_2\ \ m/sec^2}$

2-8. $\rho = 0.4$ m

$\bar\omega = \frac{10}{30}\bar e_z$



Take O' at hub of wheel.

From (2-106), $\bar a = \ddot{\bar R} + \dot{\bar\omega}\times\bar\rho + \bar\omega\times(\bar\omega\times\bar\rho) + (\ddot{\bar\rho})_r + 2\bar\omega\times(\dot{\bar\rho})_r$

$\bar R = (30 - 0.4\sin 15^\circ)\bar e_r + 0.4\cos 15^\circ\bar e_z = 29.896\bar e_r + 0.3864$

$\bar\rho = 0.4\sin 15^\circ\bar e_r + 0.4\cos 15^\circ\bar e_z = -0.1035\bar e_r + 0.3864$

$\ddot{\bar R} = -29.896\left(\frac13\right)^2\bar e_r = -3.322\bar e_r$, $\dot{\bar\omega}\times\bar\rho = 0$

$\bar\omega\times(\bar\omega\times\bar\rho) = 0.1035\left(\frac13\right)^2\bar e_r = 0.0115\bar e_r$

$(\ddot{\bar\rho})_r = \frac{10^2}{0.4}(\sin 15^\circ\bar e_r - \cos 15^\circ\bar e_z) = 64.705\bar e_r - 241.481$

since the relative velocity $(\dot{\bar\rho})_r = 10\bar e_\phi$

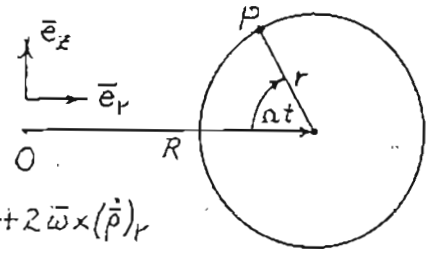
Then $2\bar\omega\times(\dot{\bar\rho})_r = -\frac{20}{3}\bar e_r$

Adding terms, we obtain

$$\underline{\bar a = 54.73\bar e_r - 241.5\bar e_z\ \ m/sec^2}$$

2-9. $\bar{R} = R\,\bar{e}_r$, $\bar{\omega} = \frac{v}{R}\bar{e}_z = \text{const.}$ $\quad \bar{e}_\phi$ is into page

$\bar{\rho} = r(-\cos\Omega t\,\bar{e}_r + \sin\Omega t\,\bar{e}_z)$



$(\dot{\bar{\rho}})_r = r\Omega(\sin\Omega t\,\bar{e}_r + \cos\Omega t\,\bar{e}_z)$

From (2-10b),

$\bar{a} = \ddot{\bar{R}} + \dot{\bar{\omega}}\times\bar{\rho} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + (\ddot{\bar{\rho}})_r + 2\bar{\omega}\times(\dot{\bar{\rho}})_r$

$\ddot{\bar{R}} = -R\omega^2\bar{e}_r = -\frac{v^2}{R}\bar{e}_r$, $\quad \dot{\bar{\omega}}\times\bar{\rho} = 0$

$\bar{\omega}\times(\bar{\omega}\times\bar{\rho}) = r\left(\frac{v}{R}\right)^2\cos\Omega t\,\bar{e}_r$, $(\ddot{\bar{\rho}})_r = r\Omega^2(\cos\Omega t\,\bar{e}_r - \sin\Omega t\,\bar{e}_z)$

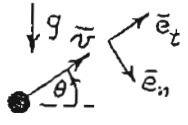
$2\bar{\omega}\times(\dot{\bar{\rho}})_r = \frac{2r}{R}v\Omega\sin\Omega t\,\bar{e}_\phi$

Adding, $\bar{a} = \left[-\frac{v^2}{R} + r\left(\frac{v^2}{R^2} + \Omega^2\right)\cos\Omega t\right]\bar{e}_r$

$+ 2\frac{r}{R}v\Omega\sin\Omega t\,\bar{e}_\phi - r\Omega^2\sin\Omega t\,\bar{e}_z$

2-10. (a) $a_n = \underline{g\cos\theta}$

(b) $a_t = \underline{-g\sin\theta}$



(c) Also $a_n = \frac{v^2}{\rho}$ or $\rho = \frac{v^2}{a_n} = \underline{\frac{v^2}{g\cos\theta}}$

(d) Also $a_n = -v\dot{\theta}$ or $\dot{\theta} = \underline{-\frac{g}{v}\cos\theta}$

(e) $\dot{\rho} = \frac{2v\dot{v}}{g\cos\theta} + \frac{v^2\dot{\theta}\sin\theta}{g\cos^2\theta}$ where $\dot{v} = a_t = -g\sin\theta$

or $\dot{\rho} = -2v\tan\theta - v\tan\theta = \underline{-3v\tan\theta}$

(f) $\ddot{\theta} = \frac{g}{v^2}\dot{v}\cos\theta + \frac{g\dot{\theta}}{v}\sin\theta = -\frac{2g^2}{v^2}\sin\theta\cos\theta$

or $\underline{\ddot{\theta} = -\frac{g^2}{v^2}\sin 2\theta}$

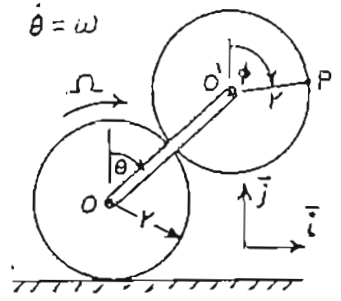
2-11.

(a) $v_{O/O'} = 2r\dot{\theta} = 2r\omega$

Also $v_{O/O'} = r\Omega + r\dot{\phi}$

i.e., the velocity of the contact point of disks relative to O plus the velocity of O' relative to the contact point. Hence

$\underline{\dot{\phi} = 2\omega - \Omega}$



(b) $\bar{a} = \bar{a}_{O'} + \bar{a}_{P/O'}$ $\quad$ Note that $\bar{a}_O = 0$.

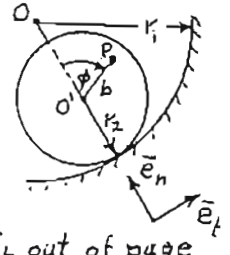
Then $\bar{a}_{O'} = 2r\omega^2(-\sin\theta\,\bar{i} - \cos\theta\,\bar{j})$

$\bar{a}_{P/O'} = r\dot{\phi}^2(-\sin\phi\,\bar{i} - \cos\phi\,\bar{j})$

$\underline{\bar{a} = \left[-2r\omega^2\sin\theta - r(2\omega-\Omega)^2\sin\phi\right]\bar{i} + \left[-2r\omega^2\cos\theta - r(2\omega-\Omega)^2\cos\phi\right]\bar{j}}$

2-12. Moving system rotates with OO'. Its angular velocity ω is found from

$v_{O'} = (r_1 - r_2)\omega = r_2(\dot{\phi} - \omega)$, $\quad \bar{\omega} = \frac{r_2}{r_1}\dot{\phi}\,\bar{e}_b$



$\bar{e}_b$ out of page

Use $\bar{a} = \ddot{\bar{R}} + \dot{\bar{\omega}}\times\bar{\rho} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + (\ddot{\bar{\rho}})_r + 2\bar{\omega}\times(\dot{\bar{\rho}})_r$

$\bar{\rho} = b(\sin\phi\,\bar{e}_t + \cos\phi\,\bar{e}_n)$

$(\dot{\bar{\rho}})_r = b\dot{\phi}(\cos\phi\,\bar{e}_t - \sin\phi\,\bar{e}_n)$

$\ddot{\bar{R}} = (r_1 - r_2)\left(\frac{r_2}{r_1}\dot{\phi}\right)^2\bar{e}_n + (r_1 - r_2)\left(\frac{r_2}{r_1}\ddot{\phi}\right)\bar{e}_t$, $\quad \dot{\bar{\omega}}\times\bar{\rho} = \frac{br_2}{r_1}\ddot{\phi}(-\cos\phi\,\bar{e}_t + \sin\phi\,\bar{e}_n)$

$\bar{\omega}\times(\bar{\omega}\times\bar{\rho}) = b\left(\frac{r_2}{r_1}\dot{\phi}\right)^2(-\sin\phi\,\bar{e}_t - \cos\phi\,\bar{e}_n)$

$(\ddot{\bar{\rho}})_r = b\ddot{\phi}(\cos\phi\,\bar{e}_t - \sin\phi\,\bar{e}_n) + b\dot{\phi}^2(-\sin\phi\,\bar{e}_t - \cos\phi\,\bar{e}_n)$

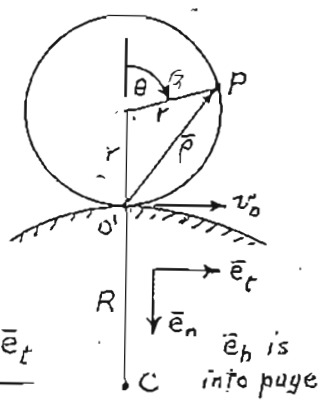
$2\bar{\omega}\times(\dot{\bar{\rho}})_r = 2\frac{br_2}{r_1}\dot{\phi}^2(\sin\phi\,\bar{e}_t + \cos\phi\,\bar{e}_n)$

Adding, and collecting terms,

$\bar{a} = \left[\left(1 - \frac{r_2}{r_1}\right)(r_2 + b\cos\phi)\ddot{\phi} - \left(1 - \frac{r_2}{r_1}\right)^2 b\dot{\phi}^2\sin\phi\right]\bar{e}_t$

$+ \left\{-\left(1 - \frac{r_2}{r_1}\right)b\ddot{\phi}\sin\phi + \left[\left(1 - \frac{r_2}{r_1}\right)\frac{r_2^2}{r_1} - \left(1 - \frac{r_2}{r_1}\right)^2 b\cos\phi\right]\dot{\phi}^2\right\}\bar{e}_n$

*13. Origin of moving frame is at O'.



$\bar\omega = \frac{v_0}{R}\bar e_b$, $\bar\rho = -r(1+\cos\theta)\bar e_n + r\sin\theta\,\bar e_t$

$\dot{\bar r} = \dot{\bar R} + (\dot{\bar\rho})_r + \bar\omega\times\bar\rho$ where $\dot{\bar R} = v_0\bar e_t$

$(\dot{\bar\rho})_r = r\dot\theta(\sin\theta\,\bar e_n + \cos\theta\,\bar e_t)$

$= v_0(\sin\theta\,\bar e_n + \cos\theta\,\bar e_t)$ since $r\dot\theta = v_0$

$\bar\omega\times\bar\rho = \frac{r}{R}v_0\left[\sin\theta\,\bar e_n + (1+\cos\theta)\bar e_t\right]$

Then $\underline{\dot{\bar r} = v_0\left(1+\frac{r}{R}\right)\sin\theta\,\bar e_n + v_0\left(1+\frac{r}{R}\right)(1+\cos\theta)\bar e_t}$

$\ddot{\bar r} = \ddot{\bar R} + \dot{\bar\omega}\times\bar\rho + \bar\omega\times(\bar\omega\times\bar\rho) + (\ddot{\bar\rho})_r + 2\bar\omega\times(\dot{\bar\rho})_r$, $\dot{\bar\omega} = -\frac{v_0}{R^2}\dot R\,\bar e_b$

$\ddot{\bar R} = \frac{v_0^2}{R}\bar e_n$; $\dot{\bar\omega}\times\bar\rho = -\frac{r\dot R}{R^2}v_0\left[\sin\theta\,\bar e_n + (1+\cos\theta)\bar e_t\right]$

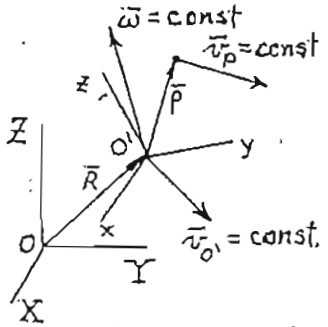
$\bar\omega\times(\bar\omega\times\bar\rho) = \frac{r}{R^2}v_0^2\left[(1+\cos\theta)\bar e_n - \sin\theta\,\bar e_t\right]$

$(\ddot{\bar\rho})_r = v_0\dot\theta(\cos\theta\,\bar e_n - \sin\theta\,\bar e_t) = \frac{v_0^2}{r}(\cos\theta\,\bar e_n - \sin\theta\,\bar e_t)$

$\bar\omega\times(\dot{\bar\rho})_r = 2\frac{v_0^2}{R}(\cos\theta\,\bar e_n - \sin\theta\,\bar e_t)$

Adding,
$$\bar a = \left[\frac{v_0^2}{r}\left(1+\frac{r}{R}\right)^2\cos\theta + \frac{v_0^2}{R}\left(1+\frac{r}{R}\right) - \frac{r v_0}{R^2}\dot R\sin\theta\right]\bar e_n - \left[\frac{v_0^2}{r}\left(1+\frac{r}{R}\right)^2\sin\theta + \frac{r v_0}{R^2}\dot R(1+\cos\theta)\right]\bar e_t$$

-14. (a) $\underline{\dot{\bar\rho} = \bar v_P - \bar v_{O'}} = \text{const}$



(b) $\dot{\bar\rho} = (\dot{\bar\rho})_r + \bar\omega\times\bar\rho$ or

$\underline{(\dot{\bar\rho})_r = \bar v_P - \bar v_{O'} - \bar\omega\times\bar\rho}$

c) $\dot{\bar v}_P = \dot{\bar v}_O = 0$ so $\underline{\ddot{\bar\rho} = \dot{\bar v}_P - \dot{\bar v}_O = 0}$

d) $\bar a_P = 0$. Using (2-106), we obtain

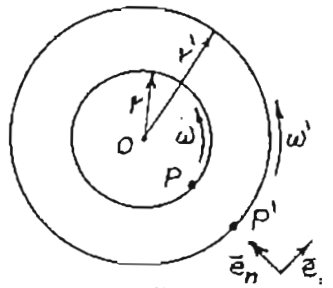
$(\ddot{\bar\rho})_r = -\bar\omega\times(\bar\omega\times\bar\rho) - 2\bar\omega\times(\dot{\bar\rho})_r = -\bar\omega\times(\bar\omega\times\bar\rho) - 2\bar\omega\times(\bar v_P - \bar v_{O'} - \bar\omega\times\bar\rho)$

$= \underline{\bar\omega\times(\bar\omega\times\bar\rho) - 2\bar\omega\times(\bar v_P - \bar v_{O'})}$

e) Using (b), then (a),

$\frac{d}{dt}\left[(\dot{\bar\rho})_r\right] = \dot{\bar v}_P - \dot{\bar v}_{O'} - \dot{\bar\omega}\times\bar\rho - \bar\omega\times\dot{\bar\rho} = \underline{-\bar\omega\times(\bar v_P - \bar v_{O'})}$

2-15. (a) Nonrotating observer.



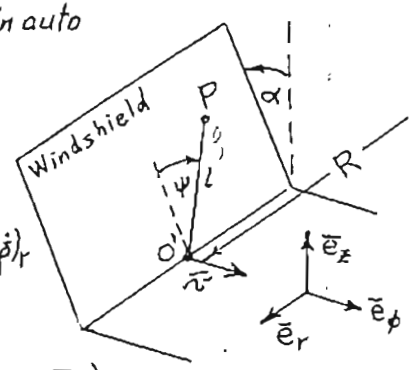
$\bar v_{P'/P} = \bar v_{P'} - \bar v_P = \underline{(r'\omega' - r\omega)\bar e_t}$

$\bar a_{P'/P} = \bar a_{P'} - \bar a_P = r'\omega'^2\bar e_n - r\omega^2\bar e_n$
$= \underline{(r'\omega'^2 - r\omega^2)\bar e_n}$

(b) Observer, rotating with unprimed system, sees the primed system rotating counterclockwise at $(\omega' - \omega)$.

$\underline{\bar v_{P'/P} = r'(\omega' - \omega)\bar e_t}$, $\underline{\bar a_{P'/P} = r'(\omega' - \omega)^2\bar e_n}$

2-16. Moving system is fixed in auto with origin at O'. $\bar\omega = \frac{v}{R}\bar e_z$



$\bar\rho = l(-\sin\psi\,\bar e_r - \cos\psi\sin\alpha\,\bar e_\phi + \cos\psi\cos\alpha\,\bar e_z)$

where $\psi = \psi_c\sin\beta t$. Use

$\bar a = \ddot{\bar R} + \dot{\bar\omega}\times\bar\rho + \bar\omega\times(\bar\omega\times\bar\rho) + (\ddot{\bar\rho})_r + 2\bar\omega\times(\dot{\bar\rho})_r$

$\ddot{\bar R} = -\frac{v^2}{R}\bar e_r$, $\dot{\bar\omega}\times\bar\rho = 0$

$\bar\omega\times(\bar\omega\times\bar\rho) = \frac{lv^2}{R^2}(\sin\psi\,\bar e_r + \cos\psi\sin\alpha\,\bar e_\phi)$

Now $(\dot{\bar\rho})_r = l\dot\psi(-\cos\psi\,\bar e_r + \sin\psi\sin\alpha\,\bar e_\phi - \sin\psi\cos\alpha\,\bar e_z)$

$2\bar\omega\times(\dot{\bar\rho})_r = \frac{2vl\dot\psi}{R}(-\sin\psi\sin\alpha\,\bar e_r - \cos\psi\,\bar e_\phi)$

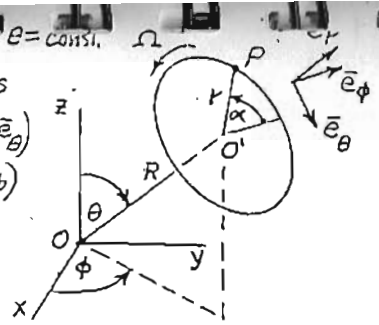
$(\ddot{\bar\rho})_r = l\ddot\psi(-\cos\psi\,\bar e_r + \sin\psi\sin\alpha\,\bar e_\phi - \sin\psi\cos\alpha\,\bar e_z)$
$+ l\dot\psi^2(\sin\psi\,\bar e_r + \cos\psi\sin\alpha\,\bar e_\phi - \cos\psi\cos\alpha\,\bar e_z)$

Adding terms, we obtain

$$\bar a = \left[-\frac{v^2}{R} + \frac{v^2}{R^2}l\sin\psi + l\dot\psi^2\sin\psi - l\ddot\psi\cos\psi - \frac{2vl\dot\psi}{R}\sin\psi\sin\alpha\right]\bar e_r$$
$$+\left[\frac{v^2}{R^2}l\cos\psi\sin\alpha + l\dot\psi^2\cos\psi\sin\alpha + l\ddot\psi\sin\psi\sin\alpha - \frac{2vl\dot\psi}{R}\cos\psi\right]\bar e_\phi$$
$$+\left[-l\ddot\psi\sin\psi\cos\alpha - l\dot\psi^2\cos\psi\cos\alpha\right]\bar e_z$$

Largest force occurs for the largest acceleration normal to windshield (outward). $\bar\omega\times(\bar\omega\times\bar\rho) + 2\bar\omega\times(\dot{\bar\rho})_r$ max. normal when $\underline{\dot\psi = -\beta\psi_c}$

2-[illegible] Giv[illegible] $\alpha = [illegible]$ const., $r = $ const., $\theta = $ const.



Moving system with origin O' rotates at $\bar{\omega} = \dot{\phi}\,\bar{e}_z$ or $\bar{\omega} = \dot{\phi}(\cos\theta\,\bar{e}_r - \sin\theta\,\bar{e}_\theta)$

Also O'P or $\bar{\rho} = r(-\sin\alpha\,\bar{e}_\theta + \cos\alpha\,\bar{e}_\phi)$

Use

$$\bar{a} = \ddot{\bar{R}} + \dot{\bar{\omega}}\times\bar{\rho} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + (\ddot{\bar{\rho}})_r + 2\bar{\omega}\times(\dot{\bar{\rho}})_r$$

$$\ddot{\bar{R}} = R\dot{\phi}^2\sin\theta\,(-\sin\theta\,\bar{e}_r - \cos\theta\,\bar{e}_\theta)$$

$$\dot{\bar{\omega}}\times\bar{\rho} = 0$$

Now

$$\bar{\omega}\times\bar{\rho} = r\dot{\phi}\begin{vmatrix} \bar{e}_r & \bar{e}_\theta & \bar{e}_\phi \\ \cos\theta & -\sin\theta & 0 \\ 0 & -\sin\alpha & \cos\alpha \end{vmatrix} = r\dot{\phi}(-\cos\alpha\sin\theta\,\bar{e}_r - \cos\alpha\cos\theta\,\bar{e}_\theta - \sin\alpha\cos\theta\,\bar{e}_\phi)$$

and we obtain

$$\bar{\omega}\times(\bar{\omega}\times\bar{\rho}) = r\dot{\phi}^2\begin{vmatrix} \bar{e}_r & \bar{e}_\theta & \bar{e}_\phi \\ \cos\theta & -\sin\theta & 0 \\ -\cos\alpha\sin\theta & -\cos\alpha\cos\theta & -\sin\alpha\cos\theta \end{vmatrix}$$

$$= r\dot{\phi}^2(\sin\alpha\sin\theta\cos\theta\,\bar{e}_r + \sin\alpha\cos^2\theta\,\bar{e}_\theta - \cos\alpha\,\bar{e}_\phi)$$

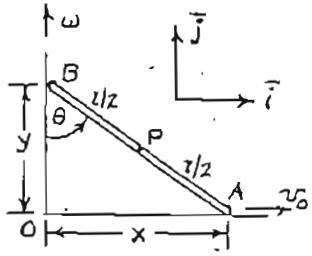
$$(\dot{\bar{\rho}})_r = r\Omega(-\cos\alpha\,\bar{e}_\theta - \sin\alpha\,\bar{e}_\phi),\quad (\ddot{\bar{\rho}}) = r\Omega^2(\sin\alpha\,\bar{e}_\theta - \cos\alpha\,\bar{e}_\phi)$$

$$2\bar{\omega}\times(\dot{\bar{\rho}})_r = 2r\Omega\dot{\phi}(\sin\alpha\sin\theta\,\bar{e}_r + \sin\alpha\cos\theta\,\bar{e}_\theta - \cos\alpha\cos\theta\,\bar{e}_\phi)$$

Adding terms,

$$\bar{a} = [-R\dot{\phi}^2\sin^2\theta + r\dot{\phi}^2\sin\alpha\sin\theta\cos\theta + 2r\Omega\dot{\phi}\sin\alpha\sin\theta]\,\bar{e}_r$$
$$+ [-R\dot{\phi}^2\sin\theta\cos\theta + r\dot{\phi}^2\sin\alpha\cos^2\theta + r\Omega^2\sin\alpha + 2r\Omega\dot{\phi}\sin\alpha\cos\theta]\,\bar{e}_\theta$$
$$+ [-r\dot{\phi}^2\cos\alpha - r\Omega^2\cos\alpha - 2r\Omega\dot{\phi}\cos\alpha\cos\theta]\,\bar{e}_\phi$$

* 2-18. Take origin of rotating system at O.



$x = l\sin\theta$, $\dot{x} = l\dot{\theta}\cos\theta$ or $\dot{\theta} = \dfrac{v_0}{l\cos\theta}$

$\bar{\omega} = \omega\bar{j} = $ const, $\bar{\rho} = \frac{l}{2}(\sin\theta\,\bar{i} + \cos\theta\,\bar{j})$

$$\bar{v} = \dot{\bar{R}} + (\dot{\bar{\rho}})_r + \bar{\omega}\times\bar{\rho}$$

$$\dot{\bar{R}} = 0,\quad (\dot{\bar{\rho}})_r = \frac{l\dot{\theta}}{2}(\cos\theta\,\bar{i} - \sin\theta\,\bar{j}) = \frac{v_0}{2\cos\theta}(\cos\theta\,\bar{i} - \sin\theta\,\bar{j})$$

$$\bar{\omega}\times\bar{\rho} = -\frac{\omega l}{2}\sin\theta\,\bar{k}$$

Hence $\bar{v} = \frac{1}{2}v_0\,\bar{i} - \frac{1}{2}v_0\tan\theta\,\bar{j} - \frac{1}{2}\omega l\sin\theta\,\bar{k}$

2-18. (cont'd.) From (2-106),

$$\bar{a} = \ddot{\bar{R}} + \dot{\bar{\omega}}\times\bar{\rho} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + (\ddot{\bar{\rho}})_r + 2\bar{\omega}\times(\dot{\bar{\rho}})_r$$

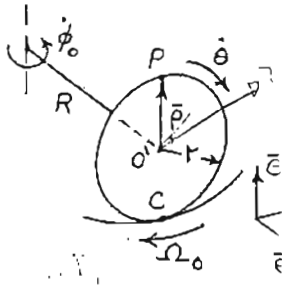
$$\ddot{\bar{R}} = 0,\quad \dot{\bar{\omega}}\times\bar{\rho} = 0,\quad \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) = -\tfrac{1}{2}l\omega^2\sin\theta\,\bar{i}$$

$$(\ddot{\bar{\rho}})_r = \frac{-v_0\dot{\theta}}{2\cos^2\theta}\bar{j} = -\frac{v_0^2}{2l\cos^3\theta}\bar{j},\quad 2\bar{\omega}\times(\dot{\bar{\rho}})_r = -\omega v_0$$

Adding,

$$\bar{a} = -\tfrac{1}{2}l\omega^2\sin\theta\,\bar{i} - \frac{v_0^2}{2l\cos^3\theta}\bar{j} - \omega v_0\,\bar{k}$$

* 2-19. Take the moving frame with the origin at O' and with $\bar{\omega} = \dot{\phi}_0\,\bar{e}_z$



$$v_{O'} - v_C = R(\dot{\phi}_0 + \Omega_0) = r\dot{\theta}$$

giving,

$$\dot{\theta} = \frac{R}{r}(\dot{\phi}_0 + \Omega_0)$$

Use

$$\bar{a} = \ddot{\bar{R}} + \dot{\bar{\omega}}\times\bar{\rho} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + (\ddot{\bar{\rho}})_r + 2\bar{\omega}\times(\dot{\bar{\rho}})_r$$

$$\ddot{\bar{R}} = -R\dot{\phi}_0^2\,\bar{e}_r,\quad \dot{\bar{\omega}}\times\bar{\rho} = 0,\quad \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) = 0$$

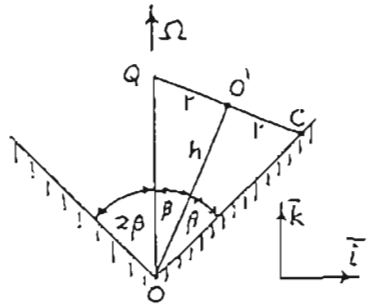
$$(\ddot{\bar{\rho}})_r = -r\dot{\theta}^2\,\bar{e}_z = -\frac{R^2}{r}(\dot{\phi}_0 + \Omega_0)^2\,\bar{e}_z$$

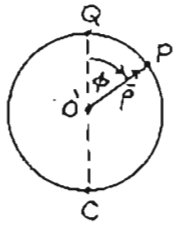
Now $(\dot{\bar{\rho}})_r = r\dot{\theta}\,\bar{e}_\phi$ so we obtain

$$2\bar{\omega}\times(\dot{\bar{\rho}})_r = -2R\dot{\phi}_0(\dot{\phi}_0 + \Omega_0)\,\bar{e}_r$$

Adding terms,

$$\bar{a} = (-3R\dot{\phi}_0^2 - 2R\dot{\phi}_0\Omega_0)\,\bar{e}_r - \frac{R^2}{r}(\dot{\phi}_0 + \Omega_0)^2\,\bar{e}_z$$





The moving system has origin at O' and angular velocity $\bar{\omega} = \Omega \bar{k}$. Relative to this frame, point Q on the cone has the velocity $v_Q = r\dot{\phi}$. But we also know that $v_Q = 2v_{O'}$ since point C is instantaneously at rest and CO'C is a diameter. Hence we can write

$$2v_{O'} = 2(r\Omega\cos\beta) = r\dot{\phi} \quad \text{or} \quad \dot{\phi} = 2\Omega\cos\beta$$

Also we see that $\bar{\rho} = r(-\cos\phi\cos\beta\,\bar{i} + \sin\phi\,\bar{j} + \cos\phi\sin\beta\,\bar{k})$.

Then we use $\bar{a} = \ddot{\bar{R}} + \dot{\bar{\omega}}\times\bar{\rho} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + (\ddot{\bar{\rho}})_r + 2\bar{\omega}\times(\dot{\bar{\rho}})_r$

$\ddot{\bar{R}} = -r\Omega^2\cos\beta\,\bar{i}$, $\dot{\bar{\omega}}\times\bar{\rho} = 0$, $\bar{\omega}\times(\bar{\omega}\times\bar{\rho}) = r\Omega^2(\cos\phi\cos\beta\,\bar{i} - \sin\phi\,\bar{j})$

Now $(\dot{\bar{\rho}})_r = r\dot{\phi}(\sin\phi\cos\beta\,\bar{i} + \cos\phi\,\bar{j} - \sin\phi\sin\beta\,\bar{k})$

$= 2r\Omega(\sin\phi\cos^2\beta\,\bar{i} + \cos\phi\cos\beta\,\bar{j} - \sin\phi\sin\beta\cos\beta\,\bar{k})$

and $(\ddot{\bar{\rho}})_r = -\dot{\phi}^2\bar{\rho} = r\dot{\phi}^2(\cos\phi\cos\beta\,\bar{i} - \sin\phi\,\bar{j} - \cos\phi\sin\beta\,\bar{k})$

$= 4r\Omega^2(\cos\phi\cos^3\beta\,\bar{i} - \sin\phi\cos^2\beta\,\bar{j} - \cos\phi\sin\beta\cos^2\beta\,\bar{k})$

$2\bar{\omega}\times(\dot{\bar{\rho}})_r = 4r\Omega^2(-\cos\phi\cos\beta\,\bar{i} + \sin\phi\cos^2\beta\,\bar{j})$

Adding, $\bar{a} = r\Omega^2\cos\beta(-1 + \cos\phi - 4\cos\phi\sin^2\beta)\,\bar{i} - r\Omega^2\sin\phi\,\bar{j}$
$- 4r\Omega^2\cos\phi\sin\beta\cos^2\beta\,\bar{k}$

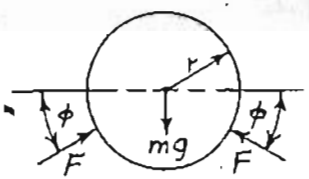
(b) For $\phi = 0$, $\bar{a} = -4r\Omega^2\sin\beta\cos\beta(\sin\beta\,\bar{i} + \cos\beta\,\bar{k})$

Acceleration magnitude $a = \frac{v^2}{\rho} = 4r\Omega^2\sin\beta\cos\beta$ where ρ = radius of curvature and $v = v_Q = 2r\Omega\cos\beta$.

Hence $\rho = \frac{v^2}{a} = r\cot\beta = h$

CHAPTER 3

3-1. The upward acceleration a is found from

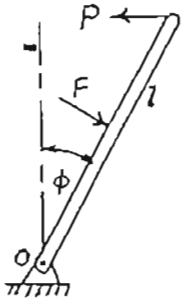


$$ma = 2F\sin\phi - mg$$

or

$$a = \frac{2F}{m}\sin\phi - g$$

Consider the massless lever. Taking moments about O,



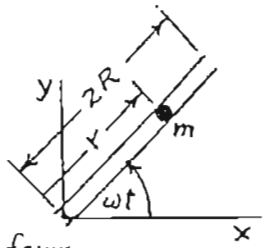
$$Fr\cot\phi - Pl\cos\phi = 0$$

giving $F = \frac{Pl}{r}\sin\phi$

Then $a = \frac{2Pl}{mr}\sin^2\phi - g$

For $\phi = 30^\circ$, $\underline{a = \frac{Pl}{2mr} - g}$

3-2. The radial acceleration is



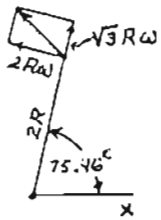
$$a_r = \ddot{r} - \omega^2 r = 0$$

since the radial force is zero.

This equation has a solution of the form

$$r = A\cosh\omega t + B\sinh\omega t \quad \text{with} \quad r(0) = R,\ \dot{r}(0) = 0$$

Hence $A = R$ and $B = C$.



$r = R\cosh\omega t$, $\dot{r} = R\omega\sinh\omega t$

When $r = 2R$, $\cosh\omega t = 2$, $\sinh\omega t = \sqrt{3}$

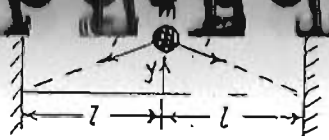
$\omega t = \cosh^{-1}2 = 1.3170$ rad $= 75.456^\circ$

Radial velocity $v_r = \dot{r} = \sqrt{3}R\omega$
Transverse velocity $v_t = 2R\omega$ $\Big\}$ $v = \sqrt{v_r^2 + v_t^2} = \sqrt{7}R\omega$

Direction (counterclockwise from x axis) $\theta = 75.46 + \tan^{-1}\frac{2}{\sqrt{3}}$
$= 124.56^\circ$

Cartesian components: $\underline{\bar{v} = (-1.5010\,\bar{i} + 2.1788)R\omega}$

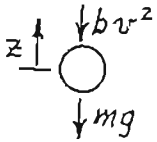
Equation of motion: $m\ddot{y} = -2T\frac{y}{l}$



or $\ddot{y} + \frac{2T}{ml}y = 0$

Natural frequency $\underline{\omega_n = \sqrt{\frac{2T}{ml}}}$

3-4. (a) Eq$^{\underline{n}}$ of motion: $m\dot{v} = -bv^2 - mg$



Integrating,

$$-\int_{v_0}^{v}\frac{dv}{g+\frac{b}{m}v^2} = t = \sqrt{\frac{m}{bg}}\left(\tan^{-1}\sqrt{\frac{b}{mg}}\,v_0 - \tan^{-1}\sqrt{\frac{b}{mg}}\,v\right)$$

Then, solving for v, $v = \sqrt{\frac{mg}{b}}\tan\sqrt{\frac{bg}{m}}\left(t^* - t\right)$

where the time of z_{max} is $t^* = \sqrt{\frac{m}{bg}}\tan^{-1}\sqrt{\frac{b}{mg}}\,v_0$

Expanding $\tan\sqrt{\frac{bg}{m}}(t^*-t)$ yields

$$\underline{v = \left(\frac{v_0 - \sqrt{\frac{mg}{b}}\tan\sqrt{\frac{bg}{m}}\,t}{1+\sqrt{\frac{b}{mg}}\,v_0\tan\sqrt{\frac{bg}{m}}\,t}\right)}$$

(b) Let $\dot{v} = v\frac{dv}{dz}$ Then $mv\frac{dv}{dz} = -bv^2 - mg$

Integrating, $z = -\int_{v_0}^{v}\frac{v\,dv}{g+\frac{b}{m}v^2} = \frac{-m}{2b}\left[\ln\left(\frac{mg}{b}+v^2\right) - \ln\left(\frac{mg}{b}+v_0^2\right)\right]$

or $e^{-\frac{2bz}{m}} = \frac{mg+bv^2}{mg+bv_0^2}$ giving $\underline{v^2 = \frac{mg}{b}\left[\left(1+\frac{bv_0^2}{mg}\right)e^{-\frac{2b}{m}z} - 1\right]}$

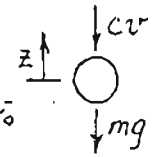
For $v=0$, we obtain $\underline{z_{max} = \frac{m}{2b}\ln\left(1+\frac{bv_0^2}{mg}\right)}$

(c) For square-law damping with $bv_0^2 = mg$, we obtain

$$\underline{z_{max} = \frac{m}{2b}\ln 2 = 0.3466\,\frac{v_0^2}{g}}$$

of motion is

$$m\dot{v} = -cv - mg \quad \text{or} \quad \dot{v} = v\frac{dv}{dz} = -\frac{c}{m}v - g$$



Integrating,

$$z_{max} = -\int_{v_0}^{0}\frac{v\,dv}{g+\frac{c}{m}v} = \frac{m^2}{c^2}\left[g+\frac{c}{m}v - g\ln\left(g+\frac{c}{m}v\right)\right]_0^{v_0}$$

$$= \frac{mv_0}{c} - \frac{m^2g}{c^2}\ln\left(1+\frac{cv_0}{mg}\right)$$

For $c = bv_0 = \frac{mg}{v_0}$, $z_{max} = \frac{v_0^2}{g} - \frac{v_0^2}{g}\ln 2 = \underline{0.3069\,\frac{v_0^2}{g}}$

Also see Eqs. (3-63) and (3-64).

3-5. $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$ where $\omega_n = \sqrt{\frac{k}{m}}$, $\zeta = \frac{c}{2\sqrt{km}}$

From (3-171), the solution has the form

$$x = Ce^{-\zeta\omega_n t}\cos(\omega_d t + \theta) \quad \text{where} \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

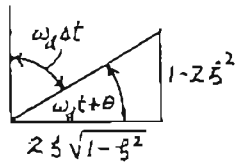
Hence $\dot{x} = Ce^{-\zeta\omega_n t}\left[-\zeta\omega_n\cos(\omega_d t+\theta) - \omega_d\sin(\omega_d t+\theta)\right]$

$|\dot{x}|_{max}$ occurs when $\ddot{x} = 0$ or, from differential equation, when $\frac{\dot{x}}{x} = -\frac{\omega_n}{2\zeta} = -\zeta\omega_n - \omega_d\tan(\omega_d t+\theta)$

Hence $\tan(\omega_d t+\theta) = \frac{1-2\zeta^2}{2\zeta\sqrt{1-\zeta^2}}$ for $|\dot{x}|_{max}$.

But $x=0$ at $\cos(\omega_d t+\theta) = 0$ or $\omega_d t + \theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots$

Thus $\tan\omega_d\Delta t = \frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}$



or $\underline{\Delta t = \frac{1}{\omega_n\sqrt{1-\zeta^2}}\tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}}$

-6. From (3-142), the weighting function is

$$h(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}} \sin \omega_d t \quad \text{where} \quad \omega_d = \omega_n\sqrt{1-\zeta^2}.$$

Using (3-195),

$$x(t) = \int_0^\infty h(\tau)F(t-\tau)\,d\tau = \frac{F_0\omega_n}{k\sqrt{1-\zeta^2}}\int_0^\infty e^{-\zeta\omega_n\tau}\sin\omega_d\tau\cos\omega(t-\tau)\,d\tau$$

where the upper limit is ∞ since the steady-state solution is desired. Next expand $\cos\omega(t-\tau)$ and obtain

$$x(t) = \frac{F_0\omega_n}{k\sqrt{1-\zeta^2}}\left\{\cos\omega t\int_0^\infty e^{-\zeta\omega_n t}\sin\omega_d\tau\cos\omega\tau\,d\tau + \sin\omega t\int_0^\infty e^{-\zeta\omega_n t}\sin\omega_d t\sin\omega\tau\,d\tau\right\}$$

Evaluating the integrals,

$$\int_0^\infty e^{-\zeta\omega_n\tau}\sin\omega_d\tau\cos\omega\tau\,d\tau = \frac{1}{2}\int_0^\infty e^{-\zeta\omega_n\tau}\left[\sin(\omega_d+\omega)\tau + \sin(\omega_d-\omega)\tau\right]d\tau$$

$$= \frac{1}{2}\left[\frac{e^{-\zeta\omega_n\tau}\left[-\zeta\omega_n\sin(\omega_d+\omega)\tau - (\omega_d+\omega)\cos(\omega_d+\omega)\tau\right]}{\zeta^2\omega_n^2+(\omega_d+\omega)^2}\right]_0^\infty$$

$$+\frac{1}{2}\left[\frac{e^{-\zeta\omega_n\tau}\left[-\zeta\omega_n\sin(\omega_d-\omega)\tau - (\omega_d-\omega)\cos(\omega_d-\omega)\tau\right]}{\zeta^2\omega_n^2+(\omega_d-\omega)^2}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{\omega_d+\omega}{\zeta^2\omega_n^2+(\omega_d+\omega)^2} + \frac{\omega_d-\omega}{\zeta^2\omega_n^2+(\omega_d-\omega)^2}\right]$$

Similarly,

$$\int_0^\infty e^{-\zeta\omega_n\tau}\sin\omega_d\tau\sin\omega\tau\,d\tau = \frac{1}{2}\int_0^\infty e^{-\zeta\omega_n\tau}\left[\cos(\omega_d-\omega)\tau - \cos(\omega_d+\omega)\tau\right]d\tau$$

$$= \frac{1}{2}\left[\frac{\zeta\omega_n}{\zeta^2\omega_n^2+(\omega_d-\omega)^2} - \frac{\zeta\omega_n}{\zeta^2\omega_n^2+(\omega_d+\omega)^2}\right]$$

Now $\omega_d = \omega_n\sqrt{1-\zeta^2}$, so we find that

$$\zeta^2\omega_n^2+(\omega_d+\omega)^2 = \zeta^2\omega_n^2+\omega_n^2(1-\zeta^2)+2\omega\omega_n\sqrt{1-\zeta^2}+\omega^2 = \omega_n^2+2\omega\omega_n\sqrt{1-\zeta^2}+\omega^2$$

$$\zeta^2\omega_n^2+(\omega_d-\omega)^2 = \omega_n^2-2\omega\omega_n\sqrt{1-\zeta^2}+\omega^2$$

Then the above integrals can be expressed in the form

3-6. (cont'd.)

$$\int_0^\infty e^{-\zeta\omega_n\tau}\sin\omega_d\tau\cos\omega\tau\,d\tau = \frac{1}{2}\left[\frac{\omega_n\sqrt{1-\zeta^2}+\omega}{\omega_n^2+\omega^2+2\omega\omega_n\sqrt{1-\zeta^2}} + \frac{\omega_n\sqrt{1-\zeta^2}-\omega}{\omega_n^2+\omega^2-2\omega\omega_n\sqrt{1-\zeta^2}}\right]$$

$$= \frac{\omega_n(\omega_n^2-\omega^2)\sqrt{1-\zeta^2}}{(\omega_n^2-\omega^2)^2+4\zeta^2\omega^2\omega_n^2}$$

$$\int_0^\infty e^{-\zeta\omega_n\tau}\sin\omega_d\tau\sin\omega\tau\,d\tau = \frac{1}{2}\left[\frac{\zeta\omega_n}{\omega_n^2+\omega^2-2\omega\omega_n\sqrt{1-\zeta^2}} - \frac{\zeta\omega_n}{\omega_n^2+\omega^2+2\omega\omega_n\sqrt{1-\zeta^2}}\right]$$

$$= \frac{2\zeta\omega\omega_n^2\sqrt{1-\zeta^2}}{(\omega_n^2-\omega^2)^2+4\zeta^2\omega^2\omega_n^2}$$

Finally,

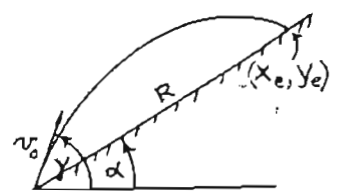
$$x(t) = \frac{F_0\omega_n}{k}\left[\frac{\omega_n(\omega_n^2-\omega^2)\cos\omega t + 2\zeta\omega\omega_n^2\sin\omega t}{(\omega_n^2-\omega^2)^2+4\zeta^2\omega^2\omega_n^2}\right]$$

$$= \frac{F_0}{k}\left[\frac{\left(1-\frac{\omega^2}{\omega_n^2}\right)\cos\omega t + 2\zeta\frac{\omega}{\omega_n}\sin\omega t}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right]$$

$$= \frac{(F_0/k)\cos(\omega t+\phi)}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \quad \text{where } \tan\phi = \frac{-2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}$$

3-7. From (3-27), the envelope of possible trajectories is

$$x_e^2 = -\frac{2v_0^2}{g}\left(y_e - \frac{v_0^2}{2g}\right)$$



and any point is a point of maximum slant range.

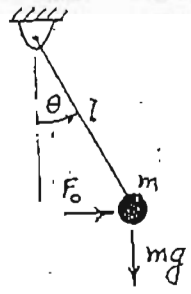
$y_e = x_e\tan\alpha$ and, from (3-26), $\frac{v_0^2}{g} = x_e\tan\gamma$

Hence $x_e^2(1+2\tan\alpha\tan\gamma-\tan^2\gamma) = 0$, $\tan\gamma = \tan\alpha+\sqrt{\tan^2\alpha+1}$
$= \tan\alpha+\sec\alpha$

where $\gamma > \alpha$ and $\frac{\pi}{4} \le \gamma < \frac{\pi}{2}$.

Then $R_{max} = \frac{x_0}{\cos\alpha} = \frac{v_0^2}{g\cos\alpha(\tan\alpha+\sec\alpha)} = \frac{v_0^2}{g(1+\sin\alpha)}$

3-8. Conservative system. Represent $F_0 = \frac{1}{2}mg$ by an additional potential energy term.



$$V = -mgl\cos\theta - \tfrac{1}{2}mgl\sin\theta$$

$$T = \tfrac{1}{2}ml^2\dot\theta^2$$

From conservation of energy,

$$\tfrac{1}{2}ml^2\dot\theta^2 - mgl\left(\cos\theta + \tfrac{1}{2}\sin\theta\right) = -mgl$$

$\dot\theta = 0$ at $\theta = \theta_{max}$, so $\cos\theta + \frac{1}{2}\sin\theta = 1$

But $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\cos\theta = 1 - \frac{1}{2}\operatorname{sen}\theta$

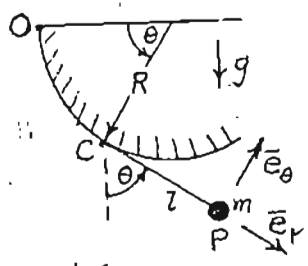
$\theta = (5^{-1} - \operatorname{tg}^{-1}\frac{1}{2})\operatorname{sen}\theta$

Hence $\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin^2\frac{\theta}{2}$

The root $\theta = 0$ represents θ_{min}, so $\tan\frac{\theta_{max}}{2} = \frac{1}{2}$, $\underline{\theta_{max} = 53.13^\circ}$

2nd Method: The two constant forces are equivalent to the force of an effective gravity of magnitude $\sqrt{5}g/2$ at $\theta = \tan^{-1}\frac{1}{2}$. The pendulum will swing equally on either side of equilibrium, giving $\theta_{max} = 2\tan^{-1}\frac{1}{2} = 53.13^\circ$.

3-9. $l = 2R - R\theta = (2-\theta)R$



Using tangential and normal components, from (2-41), $\bar a_P = \ddot s\,\bar e_\theta - \frac{\dot s^2}{\rho}\bar e_r$

where $\dot s = l\dot\theta = (2-\theta)R\dot\theta$ and $\rho = l$

$\ddot s = (2-\theta)R\ddot\theta - R\dot\theta^2$

Now $ma_\theta = F_\theta$ or $m\left[(2-\theta)R\ddot\theta - R\dot\theta^2\right] = -mg\sin\theta$

giving the differential equation $\underline{(2-\theta)R\ddot\theta - R\dot\theta^2 + g\sin\theta = 0}$

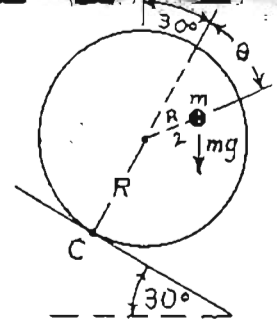
(b) Use conservation of energy. $\theta(0) = 0$, $\dot\theta(0) = \sqrt{\frac{g}{2R}}$

$$T + V = \tfrac{1}{2}m(2-\theta)^2R^2\dot\theta^2 - mgR\left[\sin\theta + (2-\theta)\cos\theta\right] = mgR - 2mgR = -mgR$$

θ_{max} occurs when $\dot\theta = 0$.

$\sin\theta + (2-\theta)\cos\theta = 1$ or $\underline{\theta_{max} = \frac{\pi}{2}}$

3-10. Conservative system. The contact point C is the instantaneous center. Thus



$$v^2 = \left[R^2 + \left(\tfrac{R}{2}\right)^2 + 2R\left(\tfrac{R}{2}\right)\cos\theta\right]\dot\theta^2$$

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}mR^2\dot\theta^2\left(\tfrac{5}{4} + \cos\theta\right)$$

$$V = -\tfrac{1}{2}mgR\theta + \tfrac{1}{2}mgR\cos(\theta + 30^\circ)$$

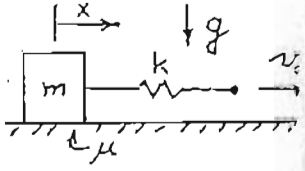
Using conservation of energy,

$$T + V = \tfrac{1}{8}mR^2\dot\theta^2(5 + 4\cos\theta) + \tfrac{1}{2}mgR\left[\cos(\theta + 30^\circ) - \theta\right] = \tfrac{\sqrt{3}}{4}mg$$

Solving for $\dot\theta$, we obtain $\underline{\dot\theta = \sqrt{\frac{g}{R}\left[\frac{2\sqrt{3} + 4\theta - 4\cos(\theta + 30^\circ)}{5 + 4\cos\theta}\right]}}$

(b) The speed of the particle is momentarily constant when a, i.e., when the tangent to the path is horizontal, and gravity do no work. $\frac{dV}{d\theta} = -\frac{1}{2}mgR - \frac{1}{2}mgR\sin(\theta + 30^\circ) = 0$, $\underline{\theta = 240^\circ}$

3-11. Sliding begins at time $t = t_1$ when the spring force equals μmg.



$kv_0t_1 = \mu mg$ or $t_1 = \frac{\mu mg}{kv_0}$

Thus $\underline{x = 0, \quad 0 \le t \le \frac{\mu mg}{kv_0}}$

For $t \ge t_1$ and $\dot x \ge 0$, we have the differential equation of motion

$$m\ddot x = k(v_0t - x) - \mu mg \quad \text{or} \quad \ddot x + \frac{k}{m}x = \frac{kv_0}{m}t - \mu g$$

Steady-state solution $x_s = C_1t + C_2$

From differential eq$^{\underline{n}}$, $\frac{k}{m}(C_1t + C_2) = \frac{kv_0}{m}t - \mu g$, $C_1 =$ $C_2 =$

Transient sol$^{\underline{n}}$ has the form $x_t = A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t$

Thus we obtain

$$x = x_s + x_t = v_0t - \frac{\mu mg}{k} + A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t$$

3-11. (cont'd.) The initial conditions at $t=t_1$ are

$x(t_1)=0$, $\dot{x}(t_1)=0$. Hence

$$x(t_1) = \frac{\mu m g}{k} - \frac{\mu m g}{k} + A\cos\frac{\mu g}{v_0}\sqrt{\frac{m}{k}} + B\sin\frac{\mu g}{v_0}\sqrt{\frac{m}{k}} = 0$$

$$\dot{x}(t_1) = v_0 - A\sqrt{\frac{k}{m}}\sin\frac{\mu g}{v_0}\sqrt{\frac{m}{k}} + B\sqrt{\frac{k}{m}}\cos\frac{\mu g}{v_0}\sqrt{\frac{m}{k}} = 0$$

Solving these equations for A and B, we obtain

$$A = v_0\sqrt{\frac{m}{k}}\sin\frac{\mu g}{v_0}\sqrt{\frac{m}{k}}, \quad B = -v_0\sqrt{\frac{m}{k}}\cos\frac{\mu g}{v_0}\sqrt{\frac{m}{k}}$$

Then

$$x = v_0\sqrt{\frac{m}{k}}\left[\sin\frac{\mu g}{v_0}\sqrt{\frac{m}{k}}\cos\sqrt{\frac{k}{m}}t - \cos\frac{\mu g}{v_0}\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t\right] + v_0 t - \frac{\mu m g}{k}$$

or

$$\underline{x = v_0\left[t - \sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}\left(t - \frac{\mu m g}{k v_0}\right)\right] - \frac{\mu m g}{k}, \quad t \geq \frac{\mu m g}{k v_0}}$$

3-12. Given that $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$, $\frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}$, $\frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}$ and

$\bar{F} = F_x\bar{i} + F_y\bar{j} + F_z\bar{k}$ which are exactness conditions,

implying that $F_x dx + F_y dy + F_z dz = -dV$, $V = V(x,y,z)$

where the minus sign was chosen arbitrarily.

Now $dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$

so $F_x = -\frac{\partial V}{\partial x}$, $F_y = -\frac{\partial V}{\partial y}$, $F_z = -\frac{\partial V}{\partial z}$ and $\bar{F} = -\nabla V$

Thus $\bar{F}\cdot d\bar{r} = -dV$ and we see that

(1) $V(x,y,z)$ is a function of position only.

(2) $\int_A^B \bar{F}\cdot d\bar{r} = -\int_A^B dV = V_A - V_B$ which is a function of the end-points only, independent of the path.

Therefore, from Sec. 3.3, $V(x,y,z)$ represents a conservative force.

3-13. Given $\bar{F} = F_x\bar{i} + F_y\bar{j} + F_z\bar{k}$ where $F_x = -x+y$, $F_y = x-y+y^2$, $F_z = 0$. First check exactness of $F_x dx + F_y dy$.

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} = 1 \quad \text{so} \quad (-x+y)dx + (x-y+y^2)dy = -dV$$

since the differential form is exact. Thus we have

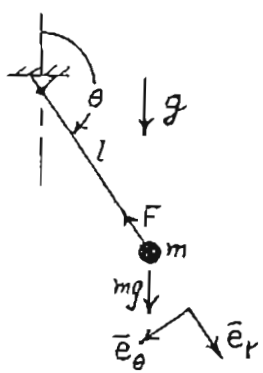
$$-\frac{\partial V}{\partial x} = -x+y \quad \text{which integrates to} \quad -V = -\tfrac{1}{2}x^2 + xy + f_1(y)$$

$$-\frac{\partial V}{\partial y} = x-y+y^2 \quad \text{which yields} \quad -V = xy - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 + f_2(x)$$

These results are consistent with

$$\underline{V = \tfrac{1}{2}x^2 - xy + \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3 + C}, \quad C = \text{arbitrary const.}$$

3-14. Using (2-33),



$$a_\phi = r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta = 0$$

since there is no force in the ϕ direction. Now $r=l$, $\dot{r}=0$, so at $t=0$,

$$\underline{\ddot{\phi}(0) = -2\dot{\theta}_0\dot{\phi}_0\cot\theta}$$

Taking θ components of force and acceleration, $mg\sin\theta = ma_\theta$ or

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta = g\sin\theta.$$

Then, at $t=0$,

$$\underline{\ddot{\theta}(0) = \dot{\phi}_0^2\sin\theta\cos\theta + \frac{g}{l}\sin\theta}$$

(b) Taking radial components of force and acceleration,

$$ma_r = -F - mg\cos\theta$$

where

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta$$

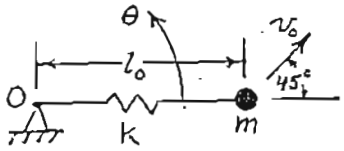
At $t=0$,

$$\underline{F = ml(\dot{\theta}_0^2 + \dot{\phi}_0^2\sin^2\theta) - mg\cos\theta}$$

3-15. Use polar coordinates.

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = \tfrac{1}{2}k(r - l_0)^2$$



Conservation of energy requires

$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \tfrac{1}{2}k(r - l_0)^2 = \tfrac{1}{2}mv_0^2$$

At max stretch, $r = 4l_0/3$, $\dot{r} = 0$, so we obtain

$$\tfrac{1}{2}m\left(\frac{4l_0}{3}\right)^2\dot{\theta}^2 + \tfrac{1}{2}k\left(\frac{l_0}{3}\right)^2 = \tfrac{1}{2}mv_0^2$$

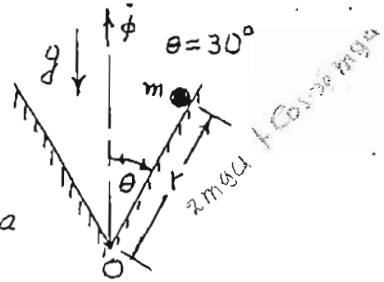
Using conservation of angular momentum,

$$mr^2\dot{\theta} = \frac{ml_0}{\sqrt{2}}v_0 \quad \text{or} \quad \dot{\theta} = \frac{l_0 v_0}{\sqrt{2}\,r^2} = \frac{9v_0}{16\sqrt{2}\,l_0} \quad \text{at } r = r_{max}$$

Then $\frac{8}{9}ml_0^2\left(\frac{9v_0}{16\sqrt{2}\,l_0}\right)^2 + \frac{kl_0^2}{18} = \frac{1}{2}mv_0^2$

or $$k = 18\frac{mv_0^2}{l_0^2}\left(\frac{1}{2} - \frac{9}{64}\right) = \frac{207\,mv_0^2}{32\,l_0^2} = 6.4688\cdot\frac{mv_0^2}{l_0^2}$$

3-16. $r(0) = a$, $\dot{r}(0) = 0$, $\dot{\phi}(0) = 4\sqrt{\frac{g}{a}}$



$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\sin^2\theta\,\dot{\phi}^2) = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{8}mr^2\dot{\phi}^2$$

$$V = mgr\cos\theta = \frac{\sqrt{3}}{2}mgr$$

Using conservation of energy,

$$\tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{8}mr^2\dot{\phi}^2 + \frac{\sqrt{3}}{2}mgr = \left(2 + \frac{\sqrt{3}}{2}\right)mga$$

Angular momentum is conserved about the vertical axis.

$$H_v = \tfrac{1}{4}mr^2\dot{\phi} = ma^{3/2}g^{1/2} \quad \text{giving} \quad \dot{\phi} = \frac{4a}{r^2}\sqrt{ag}$$

Then, from the energy eq$^{\underline{n}}$, noting that $\dot{r} = 0$ at $r = r_{max}$,

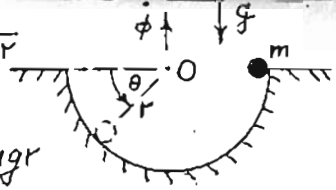
$$\frac{2mga^3}{r^2} + \frac{\sqrt{3}}{2}mgr = \left(2 + \frac{\sqrt{3}}{2}\right)mga \quad \text{or} \quad \sqrt{3}\,r^3 - (4 + \sqrt{3})ar^2 + 4a^3 = 0$$

or $(r - a)(\sqrt{3}\,r^2 - 4ar - 4a^2) = 0$ The root $r = a$ is r_{min} at $t = 0$.

$$r_{max} = \frac{2a}{\sqrt{3}}\left(1 + \sqrt{1 + \sqrt{3}}\right) = 3.0633\,a$$

3-17. $T = \tfrac{1}{2}mr^2(\dot{\theta}^2 + \dot{\phi}^2\cos^2\theta)$

$V = -mgr\sin\theta$, $v(0) = \sqrt{2gr}$



Using conservation of energy,

$$\tfrac{1}{2}mr^2(\dot{\theta}^2 + \dot{\phi}^2\cos^2\theta) - mgr\sin\theta = mgr$$

Using conservation of angular momentum about a vertical axis through the center O,

$$H_v = mr^2\cos^2\theta\,\dot{\phi} = mr\sqrt{2gr} \quad \text{or} \quad \dot{\phi} = \frac{1}{\cos^2\theta}\sqrt{\frac{2g}{r}}$$

$\dot{\theta} = 0$ at θ_{max}, so the energy equation becomes

$$\tfrac{1}{2}mr^2\left(\frac{2g}{r\cos^2\theta}\right) = mgr(1 + \sin\theta)$$

Noting that $\cos^2\theta = 1 - \sin^2\theta$, we obtain

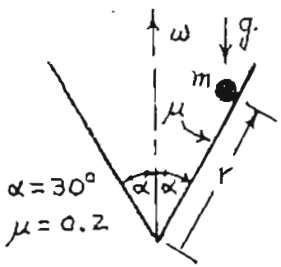
$(1 + \sin\theta)(1 - \sin^2\theta) = 1$ or $\sin\theta(\sin^2\theta + \sin\theta - 1) = 0$

$\theta_{min} = 0$, $\sin\theta_{max} = \frac{1}{2}(-1 + \sqrt{5}) = 0.6180$, $\theta_{max} = 38.17°$

3-18. Centrifugal force $= mr\omega^2\sin\alpha$

Equating normal components,

$$N = mg\sin\alpha + mr\omega^2\sin\alpha\cos\alpha$$

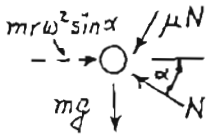


Case 1 – Sliding upward.

Consider force components in the r direction. Sliding occurs if

$$mr\omega^2\sin^2\alpha - mg\cos\alpha > \mu N$$

or $$\omega^2 > \frac{g}{r}\left(\frac{\mu + \cot\alpha}{\sin\alpha - \mu\cos\alpha}\right)$$



sliding up

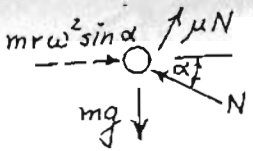
Hence ω_{max} for no sliding is

$$\omega_{max} = \sqrt{\frac{g(\mu + \cot\alpha)}{r(\sin\alpha - \mu\cos\alpha)}} = 2.4315\sqrt{\frac{g}{r}}$$

Case 2 – Sliding downward. The forces are the same as in Case 1, except that the friction force μN is reversed.

3-18. (cont'd.) Thus we obtain

$$mr\omega^2 \sin^2\alpha + \mu N < mg\cos\alpha$$



or

$$\omega^2 < \frac{g}{r}\left(\frac{\cot\alpha - \mu}{\sin\alpha + \mu\cos\alpha}\right)$$

sliding down

if sliding downward occurs.

$$\omega_{min} = \sqrt{\frac{g(\cot\alpha - \mu)}{r(\sin\alpha + \mu\cos\alpha)}} = 1.5086\sqrt{\frac{g}{r}}$$

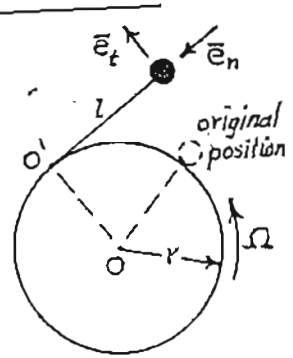
These results are also obtained by noting that incipient sliding occurs when the resultant of the centrifugal and gravitational forces lies on the cone of friction.

(b) The cone of friction angle ϕ is given by $\tan\phi = \mu$

If $\phi > \alpha$, or $\mu > \tan\alpha$, then the resultant of N and μN has a downward component. Gravity also is downward. Thus, no amount of centrifugal force will cause the particle to slide upward. Also, the expression for ω_{max}, Case 1, becomes imaginary.

$$\mu_{min} = \tan\alpha = 1/\sqrt{3}$$

3-19. $l = r\theta$, $\dot{l} = r\dot{\theta}$ or $\dot{\theta} = \frac{\dot{l}}{r}$



Choose the moving system with origin at O' and rotating at $\omega = \Omega + \dot{\theta} = \Omega + \frac{\dot{l}}{r}$.

Using (2-106),

$$\bar{a} = \ddot{\bar{R}} + \dot{\bar{\omega}}\times\bar{\rho} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + (\ddot{\bar{\rho}})_r + 2\bar{\omega}\times(\dot{\bar{\rho}})_r$$

$$R = r\bar{e}_t, \quad \bar{\rho} = -l\bar{e}_n, \quad (\dot{\bar{\rho}})_r = -\dot{l}\,\bar{e}_n$$

$$\bar{\omega} = \left(\Omega + \frac{\dot{l}}{r}\right)\bar{e}_b \text{ (out of page)}, \quad \dot{\bar{\omega}} = \frac{\ddot{l}}{r}\bar{e}_b$$

3-19. (cont'd.) $\ddot{\bar{R}} = r\dot{\omega}\,\bar{e}_n - r\omega^2\,\bar{e}_t = \ddot{l}\,\bar{e}_n - r\left(\Omega + \frac{\dot{l}}{r}\right)^2\bar{e}_t$

$$\dot{\bar{\omega}}\times\bar{\rho} = \frac{l\ddot{l}}{r}\bar{e}_t, \quad \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) = l\left(\Omega + \frac{\dot{l}}{r}\right)^2\bar{e}_n$$

$$(\ddot{\bar{\rho}})_r = -\ddot{l}\,\bar{e}_n, \quad 2\bar{\omega}\times(\dot{\bar{\rho}})_r = 2\dot{l}\left(\Omega + \frac{\dot{l}}{r}\right)\bar{e}_t$$

Adding, $\bar{a} = \left[-r\left(\Omega + \frac{\dot{l}}{r}\right)^2 + \frac{l\ddot{l}}{r} + 2\dot{l}\left(\Omega + \frac{\dot{l}}{r}\right)\right]\bar{e}_t + l\left(\Omega + \frac{\dot{l}}{r}\right)^2\bar{e}_n$

There is no force on the particle in the $\bar{e}_t$ direction, so

$$a_t = 0 \text{ or } -\frac{1}{r}(\dot{l}^2 + 2r\Omega l + r^2\Omega^2) + \frac{l\ddot{l}}{r} + \frac{1}{r}(2\dot{l}^2 + 2r\Omega l) = 0$$

giving the differential equation

$$l\ddot{l} + \dot{l}^2 - r^2\Omega^2 = 0 \text{ or } \frac{d}{dt}(l\dot{l}) = r^2\Omega^2$$

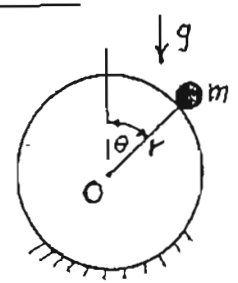
Initial conditions: $l(0) = 0$, $\dot{l}(0) = r\Omega$. Integration of the differential equation yields $l\dot{l} = r^2\Omega^2 t$. A second integration results in

$$\int l\,dl = \int r^2\Omega^2 t\,dt \text{ or } \frac{1}{2}l^2 = \frac{1}{2}r^2\Omega^2 t^2, \text{ so } l = r\Omega t$$

Tensile force $F = ma_n = ml\left(\Omega + \frac{\dot{l}}{r}\right)^2 = 4ml\Omega^2$

But $l = r\Omega t$ so we obtain $F = 4mr\Omega^3 t$

3-20. $T = \frac{1}{2}mr^2\dot{\theta}^2$, $V = mgr\cos\theta$

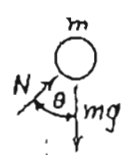


Using conservation of energy,

$$\frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta = mgr, \quad \dot{\theta}^2 = \frac{2g}{r}(1 - \cos\theta)$$

Now $ma_r = F_r$ or $-mr\dot{\theta}^2 = N - mg\cos\theta$

resulting in $N = mg(3\cos - 2)$

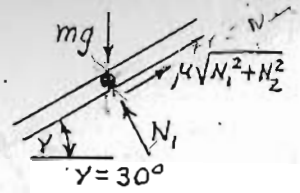


The particle will leave the sphere when the normal force N decreases to zero.

$$\cos\theta = \frac{2}{3} \text{ or } \theta = 48.19°$$

3-21. Normal force $N = \sqrt{N_1^2 + N_2^2}$

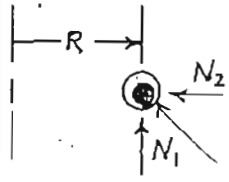
For constant speed, the acceleration is centripetal (horizontal).



$$N_2 = ma = \frac{m}{R}\left(\frac{\sqrt{3}}{2}v\right)^2$$

Balancing forces in the N_1 direction,

$$N_1 = \frac{\sqrt{3}}{2}mg$$



View looking upward into tube.

Friction force $F_f = \mu N = \mu\sqrt{N_1^2 + N_2^2}$

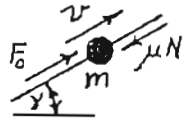
Balancing forces along tube,

$$\frac{1}{2}mg = \mu\sqrt{N_1^2+N_2^2} \quad \text{or} \quad N_2^2 = \left(\frac{mg}{2\mu}\right)^2 - \frac{3}{4}m^2g^2$$

Equating expressions for N_2,

$$N_2 = \frac{3mv^2}{4R} = \frac{1}{2}mg\sqrt{\frac{1}{\mu^2}-3} \quad \text{giving} \quad \underline{v = \sqrt{\frac{2Rg}{3}}\left(\frac{1}{\mu^2}-3\right)^{1/4}}$$

3-22. The normal force N is directed radially inward, perpendicular to the axis.



$$N = \frac{m(v\cos\gamma)^2}{R}$$

Taking tangential components of force and acceleration,

$$m\dot{v} = F_0 - \frac{\mu m}{R}v^2\cos^2\gamma \quad \text{where } \dot{v} = v\frac{dv}{ds}$$

Then $v\dfrac{dv}{ds} + \dfrac{\mu\cos^2\gamma}{R}v^2 = \dfrac{F_0}{m}$ or $\dfrac{d(v^2)}{ds} + \dfrac{2\mu\cos^2\gamma}{R}v^2 = \dfrac{2F_0}{m}$

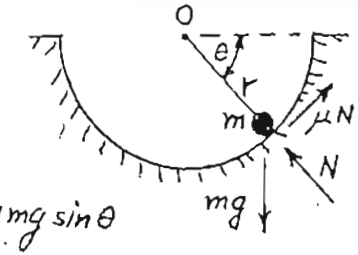
which is linear in v^2. The form of the solution is

$$v^2 = Ce^{-\frac{2\mu\cos^2\gamma}{R}s} + \frac{F_0R}{\mu m\cos^2\gamma} \quad \text{where } v = 0 \text{ at } s = 0$$

$$C = \frac{-F_0R}{\mu m\cos^2\gamma}$$

$$\underline{v^2 = \frac{F_0R}{\mu m\cos^2\gamma}\left(1 - e^{-\frac{2\mu\cos^2\gamma}{R}s}\right)}$$

3-23. The radial equation is



$$mr\dot{\theta}^2 = N - mg\sin\theta$$

giving

$$N = mr\dot{\theta}^2 + mg\sin\theta$$

θ eq$^{\underline{n}}$:

$$mr\ddot{\theta} = mg\cos\theta - \mu N$$
$$= mg\cos\theta - \mu mr\dot{\theta}^2 - \mu mg\sin\theta$$

Thus, the differential equation is

$$\underline{\ddot{\theta} + \mu\dot{\theta}^2 + \frac{g}{r}(\mu\sin\theta - \cos\theta) = 0}$$

(b) Change independent variable to θ. $\ddot{\theta} = \dot{\theta}\dfrac{d\dot{\theta}}{d\theta} = \dfrac{1}{2}\dfrac{d(\dot{\theta}^2)}{d\theta}$

Then $\dfrac{d\dot{\theta}^2}{d\theta} + 2\mu\dot{\theta}^2 = \dfrac{2g}{r}(\cos\theta - \mu\sin\theta)$

which is linear in $\dot{\theta}^2$. The steady-state solution has the form $(\dot{\theta}^2)_s = C_1\cos\theta + C_2\sin\theta$

Substituting into the differential equation,

$$(C_2 + 2\mu C_1)\cos\theta + (-C_1 + 2\mu C_2)\sin\theta = \frac{2g}{r}\cos\theta - \frac{2\mu g}{r}\sin\theta$$

Comparing coefficients of $\cos\theta$ and of $\sin\theta$,

$$\left.\begin{aligned} 2\mu C_1 + C_2 &= \frac{2g}{r} \\ -C_1 + 2\mu C_2 &= -\frac{2\mu g}{r} \end{aligned}\right\} \text{solving,} \quad \begin{aligned} C_1 &= \frac{6\mu g}{r(1+4\mu^2)} \\ C_2 &= \frac{2g}{r}\left(\frac{1-2\mu^2}{1+4\mu^2}\right) \end{aligned}$$

The transient solution has the form

$$(\dot{\theta}^2)_t = Ae^{-2\mu\theta}$$

Initial condition $\dot{\theta}(0) = 0$ so $A + C_1 = 0$, $A = \dfrac{-6\mu g}{r(1+4\mu^2)}$

Finally,

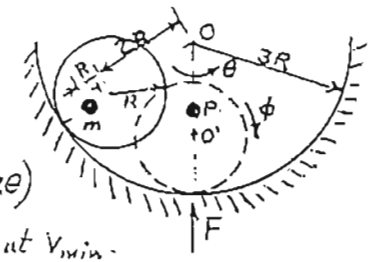
$$\underline{\dot{\theta}^2 = \frac{-6\mu g\, e^{-2\mu\theta}}{r(1+4\mu^2)} + \frac{2g}{r(1+4\mu^2)}\left[3\mu\cos\theta + (1-2\mu^2)\sin\theta\right]}$$

3-24. $v_{O'} = 2R\dot{\theta} = R\dot{\phi}$

so $\dot{\phi} = 2\dot{\theta}$. Let $\phi = 2\theta$

with $\phi = 0$, $\theta = 0$ at the lowest position. $T = \frac{1}{2} m v^2$

$$V = mgR\left(-2\cos\theta + \tfrac{1}{4}\cos 2\theta\right)$$



Conservative system. T_{max} occurs at V_{min}.

$\frac{dV}{d\theta} = mgR\left(2\sin\theta - \frac{1}{2}\sin 2\theta\right) = 0$ giving $V_{min} = -\frac{7}{4}mgR$ at $\theta = 0$.

Now $V(0) = -\frac{9}{8}mgR$ at $\theta = -60°$.

Using conservation of energy, $\frac{1}{2}mv^2 - \frac{7}{4}mgR = -\frac{9}{8}mgR$

$$v^2 = \frac{5}{4}gR, \quad \underline{v_{max} = \frac{1}{2}\sqrt{5gR}}$$

(b) When $\theta = 0$, $\phi = 0$ we see that $\dot{\phi} = \frac{v_{max}}{\frac{5}{4}R} = 2\sqrt{\frac{g}{5R}}$

$$\dot{\theta} = \frac{1}{2}\dot{\phi} = \sqrt{\frac{g}{5R}}$$

$a_{O'} = 2R\dot{\theta}^2 = \frac{2}{5}g$ upward. $a_{P/O'} = \frac{1}{4}R\dot{\phi}^2 = \frac{1}{5}g$ downward

$a_P = a_{O'} + a_{P/O'} = \frac{1}{5}g$ upward. $F - mg = m\left(\frac{1}{5}g\right)$ or $\underline{F = \frac{6}{5}mg}$

3-25. Let $\dot{\phi} = \omega = \text{const.}$

Transverse acceleration $a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi} = 2r\omega$

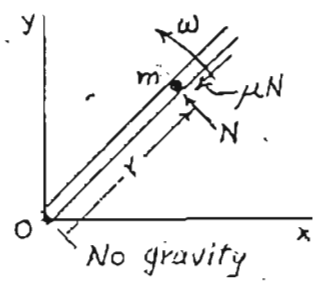
Hence $N = ma_\phi = 2m\omega\dot{r}$

Radial acceleration $a_r = \ddot{r} - r\omega^2$

$F_r = ma_r$ or $-\mu N = m(\ddot{r} - r\omega^2)$

giving the differential equation

$$\underline{\ddot{r} + 2\mu\omega\dot{r} - \omega^2 r = 0}$$



Assume solutions of form $e^{\lambda t}$. The characteristic equation is $\lambda^2 + 2\mu\omega\lambda - \omega^2 = 0$ with roots $\lambda_{1,2} = \left(-\mu \pm \sqrt{1+\mu^2}\right)\omega$

3-25. (cont'd.) The solution is of the form

$$r = e^{-\mu\omega t}\left(A\cosh\sqrt{1+\mu^2}\,\omega t + B\sinh\sqrt{1+\mu^2}\,\omega t\right)$$

Given $r(0) = R$, so $A = R$. Also, $\dot{r}(0) = 0 = -\mu A + \sqrt{1+\mu^2}\,B$

yielding $B = \frac{\mu R}{\sqrt{1+\mu^2}}$. Hence we obtain

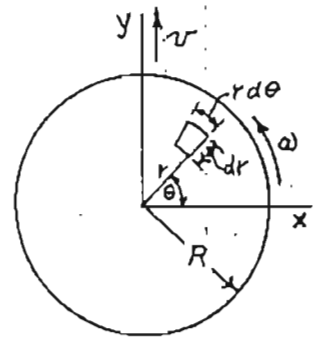
$$\underline{r(t) = Re^{-\mu\omega t}\left(\cosh\sqrt{1+\mu^2}\,\omega t + \frac{\mu}{\sqrt{1+\mu^2}}\sinh\sqrt{1+\mu^2}\,\omega t\right)}$$

Alternate form is

$$r(t) = \frac{1}{2}Re^{-\mu\omega t}\left[\left(1 + \frac{\mu}{\sqrt{1+\mu^2}}\right)e^{\sqrt{1+\mu^2}\,\omega t} + \left(1 - \frac{\mu}{\sqrt{1+\mu^2}}\right)e^{-\sqrt{1+\mu^2}\,\omega t}\right]$$

3-26. $\bar{v} = v\bar{j}$ where $v \ll R\omega$.

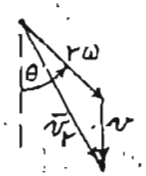
At an area element $dA = r\,dr\,d\theta$ of the shaft, the relative velocity of the plate is $\bar{v}_r = r\omega\sin\theta\,\bar{i} - (r\omega\cos\theta + v)\bar{j}$



The friction stress $\tau_f = \mu N/\pi R^2$ acts on the shaft in the direction of $\bar{v}_r$.

Hence the total frictional force is

$$\bar{F}_f = \frac{\mu N}{\pi R^2}\int_0^{2\pi}\int_0^R \left[\frac{\sin\theta\,\bar{i} - \left(\cos\theta + \frac{v}{r\omega}\right)\bar{j}}{\sqrt{1 + \frac{2v}{r\omega}\cos\theta + \frac{v^2}{r^2\omega^2}}}\right] r\,dr\,d\theta$$



By symmetry, the x component of $\bar{F}_f$ vanishes.

For $\frac{v}{r\omega} \ll 1$, we can use the approximation

$$\left[1 + \frac{2v}{r\omega}\cos\theta + \left(\frac{v}{r\omega}\right)^2\right]^{-1/2} \cong 1 - \frac{v}{r\omega}\cos\theta$$

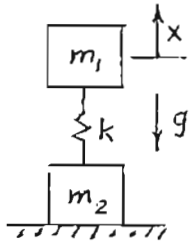
Then, to first order in $v/r\omega$, we obtain

$$\bar{F}_f = \frac{-\mu N}{\pi R^2}\int_0^R\int_0^{2\pi} r\left(\cos\theta + \frac{v}{r\omega}\sin^2\theta\right)\bar{j}\,d\theta\,dr = \frac{-\mu N}{\pi R^2}\int_0^R \frac{v\pi}{r\omega}\bar{j}\,r\,dr = \underline{\frac{-\mu N}{R\omega}\bar{v}}$$

Note: We were given that $\frac{v}{R\omega} = \epsilon \ll 1$. But we assumed that $\frac{v}{r\omega} \ll 1$. This last assumption is clearly not valid in a circle of radius r where $r/R = \epsilon$. But the frictional force on this small circle is, at most, $\epsilon^2\mu N$ compared to the total frictional force $\epsilon\mu N$, giving a negligible error.

CHAPTER 4

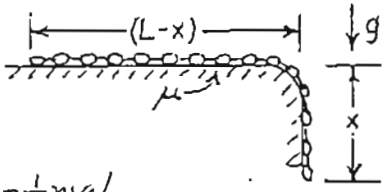
4-1. Let x be measured from the unstressed position. m_2 leaves the floor if $Kx_{max} > m_2 g$. The static deflection $x_{st} = -\frac{1}{K}(m_1 g)$ is the average value of x in the sinusoidal oscillations of m_1. In the borderline case,



$$x_{st} = \frac{1}{2}(x_{max} + x_{min}) \text{ or } \frac{-1}{K} m_1 g = \frac{1}{2}\left(\frac{m_2 g}{k} - \delta\right)$$

since $\delta = -x_{min}$. Thus we obtain $\underline{\delta = \frac{1}{K}(2m_1 + m_2)g}$

4-2. Linear density $\rho = \frac{m}{L}$.



$x(0) = a$. The initial potential energy $V_i = -\frac{1}{2}\rho a^2 = \frac{-1}{2L} mga^2$

The final potential energy $V_f = -\frac{1}{2} mgL$.

Initial kinetic energy $T_i = 0$. Final kinetic energy $T_f = \frac{1}{2} m v^2$.

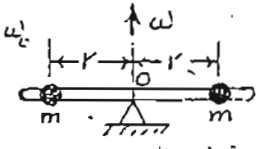
Energy lost in friction $W_f = \int_a^L \frac{\mu mg}{L}(L-x)\,dx = \frac{\mu mg}{2L}(L-a)^2$

From the work and energy principle of (4-26),

$$T_i + V_i = T_f + V_f + W_f \text{ or } -\frac{mga^2}{2L} = \frac{1}{2} m v^2 - \frac{1}{2} mgL + \frac{\mu mg}{2L}(L-a)^2$$

$$\underline{v = \sqrt{\frac{g}{L}\left[L^2 - a^2 - \mu(L-a)^2\right]}}$$

4-3. Given $r = r_0 + A\sin\beta t$, $A < r_0$, $\omega(0) = \omega_0$.



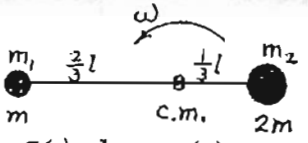
Conservation of angular momentum about O yields

$$H = 2mr^2\omega = 2mr_0^2\omega_0 \text{ or } \omega = \frac{r_0^2}{r^2}\omega_0$$

Radial acceleration $a_r = \ddot{r} - r\omega^2 = -A\beta^2 \sin\beta t - \frac{r_0^4\omega_0^2}{r^3}$

Radial force $F_r = ma_r = -m\left[A\beta^2 \sin\beta t + \frac{r_0^4\omega_0^2}{(r_0 + A\sin\beta t)^3}\right]$

4-4. Using conservation of angular momentum about the c.m.,



$$\left[m\left(\tfrac{2}{3}l\right)^2 + 2m\left(\tfrac{1}{3}l\right)^2\right]\omega = \frac{2}{3} m l_0^2 \omega_0$$

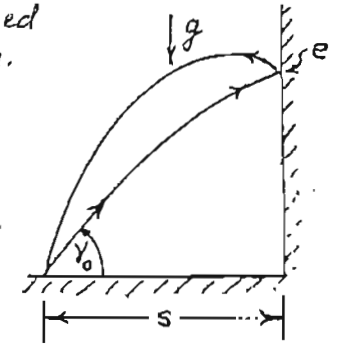
$l(0) = l_0$, $\omega(0) = \omega_0$

When $l = \frac{1}{2} l_0$, we have $\frac{1}{6} m l_0^2 \omega = \frac{2}{3} m l_0^2 \omega_0$ or $\underline{\omega = 4\omega_0}$

(b) The c.m. is fixed in an inertial frame. The string tension is

$$F = m_1 \frac{l_0}{3}\omega^2 = \frac{m l_0}{3}(4\omega_0)^2 = \underline{\frac{16}{3} m l_0 \omega_0^2}$$

4-5. The vertical velocity is unchanged upon impact because there no friction.



Total time $t = \frac{2v_0 \sin\gamma_0}{g}$

Time to reach wall $t_1 = \frac{s}{v_0 \cos\gamma_0}$

Time to return $t_2 = \frac{1}{e} t_1$

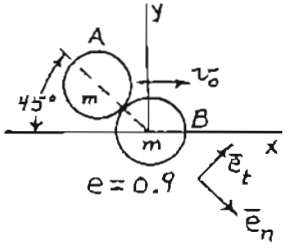
since its horizontal velocity is $ev_0 \cos\gamma_0$.

Then $t = t_1 + t_2 = \left(1 + \frac{1}{e}\right)\frac{s}{v_0 \cos\gamma_0} = \frac{2v_0 \sin\gamma_0}{g}$

Solving for γ_0, we obtain $\underline{\gamma_0 = \frac{1}{2}\sin^{-1}\left[\left(1 + \frac{1}{e}\right)\frac{gs}{v_0^2}\right]}$

for $v_0^2 \geq \left(1 + \frac{1}{e}\right)gs$. If the inequality applies, there are two values of γ_0, one in the range $0 < \gamma_0 < 45°$; the other $45° < \gamma_0 < 90°$.

4-6. Before impact,



$$v_{nA} = v_{tA} = \frac{v_0}{\sqrt{2}}, \quad v_{nB} = v_{tB} = 0$$

The tangential velocities are unchanged by the impact.

$$v'_{tA} = \frac{v_0}{\sqrt{2}}, \quad v'_{tB} = 0$$

-6. (cont'd.) Using (4-83), the normal velocities after impact are

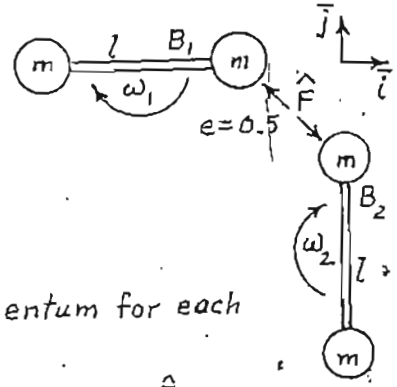
$$v'_{nA} = \left(\frac{1-e}{2}\right)v_{nA} + \left(\frac{1+e}{2}\right)v_{nB} = \frac{v_0}{20\sqrt{2}}$$

$$v'_{nB} = \left(\frac{1+e}{2}\right)v_{nA} + \left(\frac{1-e}{2}\right)v_{nB} = \frac{19 v_0}{20\sqrt{2}}$$

Taking $\bar{i}$ and $\bar{j}$ components,

$$\bar{v}'_A = v_0\left[\left(\tfrac{1}{2}+\tfrac{1}{40}\right)\bar{i} + \left(\tfrac{1}{2}-\tfrac{1}{40}\right)\bar{j}\right] = \underline{v_0(0.525\bar{i}+0.475\bar{j})}$$

$$\bar{v}'_B = v_0\left[\tfrac{19}{40}\bar{i} - \tfrac{19}{40}\bar{j}\right] = \underline{v_0(0.475\bar{i}-0.475\bar{j})}$$



4-7. Using linear impulse and momentum, the c.m. velocities are

$$\bar{v}'_1 = \left(v_0 - \frac{\hat{F}}{2\sqrt{2}\,m}\right)\bar{i} + \frac{\hat{F}}{2\sqrt{2}\,m}\bar{j}$$

$$\bar{v}'_2 = \frac{\hat{F}}{2\sqrt{2}\,m}\bar{i} - \frac{\hat{F}}{2\sqrt{2}\,m}\bar{j}$$

Using angular impulse and momentum for each dumbbell about its center,

$$\hat{M}_1 = -\frac{\hat{F}l}{2\sqrt{2}} = 2m\left(\frac{l}{2}\right)^2\omega'_1 \quad \text{or} \quad \omega'_1 = \frac{-\hat{F}}{\sqrt{2}\,ml}$$

$$\hat{M}_2 = \frac{\hat{F}l}{2\sqrt{2}} = 2m\left(\frac{l}{2}\right)^2\omega'_2 \quad \text{or} \quad \omega'_2 = \frac{\hat{F}}{\sqrt{2}\,ml}$$

From the definition of e,

$$e\frac{v_0}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\frac{\hat{F}}{\sqrt{2}\,m} - v_0\right) + \frac{1}{\sqrt{2}}\left(\frac{\hat{F}}{\sqrt{2}\,m}\right) + \frac{1}{\sqrt{2}}\left(\frac{l}{2}\right)\left(\frac{2\hat{F}}{\sqrt{2}\,ml}\right)$$

or

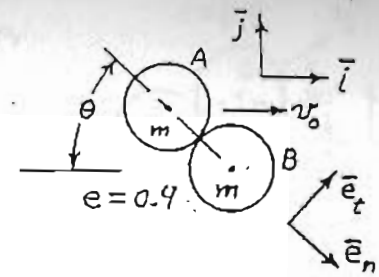
$$\frac{v_0}{2\sqrt{2}} = -\frac{v_0}{\sqrt{2}} + \frac{3\hat{F}}{2m} \quad \text{giving} \quad \hat{F} = \frac{mv_0}{\sqrt{2}}$$

Then $\underline{\bar{v}'_1 = \frac{v_0}{4}(3\bar{i}+\bar{j})}$, $\underline{\bar{v}'_2 = \frac{v_0}{4}(\bar{i}-\bar{j})}$

$\underline{\omega'_1 = -\frac{v_0}{2l}}$, $\underline{\omega'_2 = \frac{v_0}{2l}}$

4-8. $v_{nA} = v_0\cos\theta$

$v_{tA} = v_0\sin\theta$



$$v'_{nA} = \frac{1-e}{2}v_{nA} = \frac{v_0}{20}\cos\theta$$

$$v'_{tA} = v_{tA} = v_0\sin\theta$$

Shifting to $\bar{i}, \bar{j}$ unit vectors,

$$\bar{v}'_A = v_0\left[\left(\tfrac{1}{20}\cos^2\theta + \sin^2\theta\right)\bar{i} + \left(-\tfrac{1}{20}\sin\theta\cos\theta + \sin\theta\cos\theta\right)\bar{j}\right]$$

Final velocity $\bar{v}'_A$ has $\bar{i}$ and $\bar{j}$ components equal.

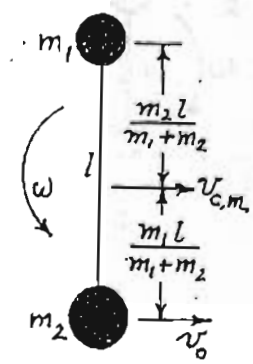
$$\frac{1}{20}\cos^2\theta + \sin^2\theta = \frac{19}{20}\sin\theta\cos\theta \quad \text{or} \quad \tan^2\theta - \frac{19}{20}\tan\theta + \frac{1}{20} = 0$$

$$\tan\theta = \frac{19}{40} \pm \sqrt{\left(\frac{19}{40}\right)^2 - \frac{1}{20}} = 0.05542,\ 0.8941 \qquad \underline{\theta = 3.20^\circ,\ 41.80^\circ}$$

4-9. The initial velocity of m_2 is v_0, due to the impulse $\hat{F} = m_2 v_0$. The initial velocity of m_1 is zero.

After the initial impulse, the velocity of the c.m. is constant and is found from

$$\hat{F} = (m_1+m_2)v_{c.m.} \quad \text{or} \quad \underline{v_{c.m.} = \frac{m_2 v_0}{m_1+m_2}}$$



After the impulse, the angular momentum is constant, so the angular velocity is constant at its initial value. $\underline{\omega = \frac{v_0}{l}}$

Consider the rotational motion about the c.m. The tensile force is

$$F = m_2\left(\frac{m_1 l}{m_1+m_2}\right)\omega^2 = \underline{\frac{m_1 m_2 v_0^2}{(m_1+m_2)l}}$$